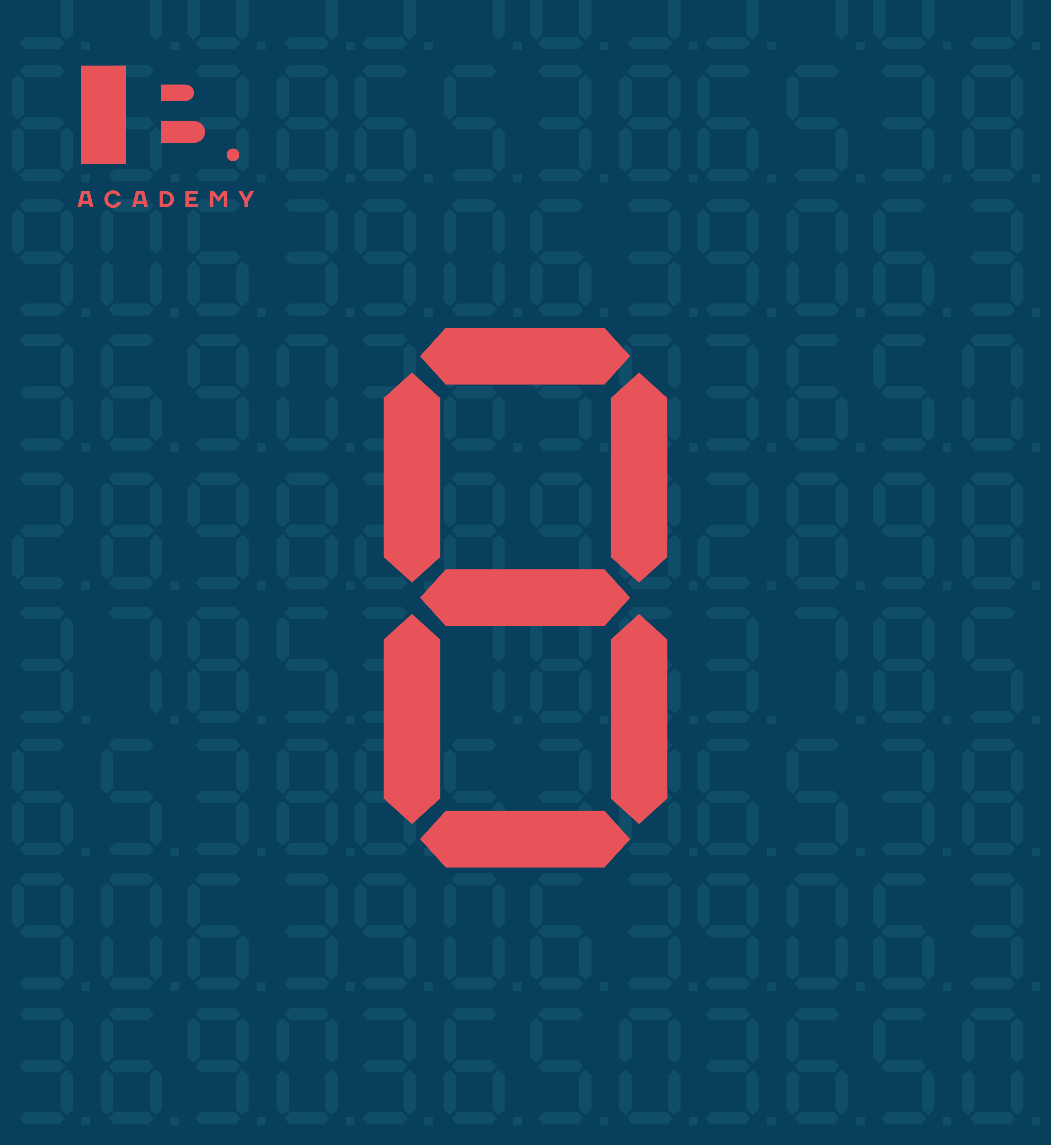

STUDY GUIDE

MATH AI HL

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ACADEMY

Mathematics Applications and Interpretations HL Study Guide

Available on www.ib.academy

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Welcome to the IB.Academy guide for Mathematics Applications and Interpretations HL.

Our Study Guides are put together by our teachers who worked tirelessly with students and schools. The idea is to compile revision material that would be easy-to-follow for IB students worldwide and for school teachers to utilise them for their classrooms. Our approach is straightforward: by adopting a step-by-step perspective, students can easily absorb dense information in a quick and efficient manner. With this format, students will be able to tackle every question swiftly and without any difficulties.

For this guide, we supplement the new topics with relevant sections from our previous Math Studies, SL and HL study resources, and with insights from our years of experience teaching these courses. We illustrate theoretical concepts by working through IB-style questions and break things down using a step-by-step approach. We also include detailed instructions on how to use the TI-Nspire™ to solve problems; most of this is also quite easily transferable to other GDC models.

The best way to apply what you have learned from the guides is with a study partner. We suggest revising with a friend or with a group in order to immediately test the information you gathered from our guides. This will help you not only process the information, but also help you formulate your answers for the exams. Practice makes better and what better way to do it than with your friends!

In order to maintain our Study Guides and to put forth the best possible material, we are in constant collaboration with students and teachers alike. To help us, we ask that you provide feedback and suggestions so that we can modify the contents to be relevant for IB studies. We appreciate any comments and hope that our Study Guides will help you with your revision or in your lessons. For more information on our material or courses, be sure to check our site at www.ib.academy.

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ALGEBRA

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Arithmetic: +/− common difference

$$u_n = n^{\text{th}} \text{ term} = u_1 + (n-1)d$$
$$S_n = \text{sum of } n \text{ terms} = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

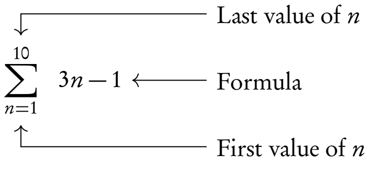
with $u_1 = a = 1^{\text{st}}$ term, $d =$ common difference.

Geometric: ×/÷ common ratio

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1}$$
$$S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)}$$
$$S_\infty = \text{sum to infinity} = \frac{u_1}{1-r}, \text{ when } -1 < r < 1$$

with $u_1 = a = 1^{\text{st}}$ term, $r =$ common ratio.

Sigma notation
A shorthand to show the sum of a number of terms in a sequence.



e.g.

$$\sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot1)-1}_{n=1} + \underbrace{(3\cdot2)-1}_{n=2} + \cdots + \underbrace{(3\cdot10)-1}_{n=10} = 155$$

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Exponents

$$x^1 = x \qquad x^0 = 1$$
$$x^m \cdot x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$
$$(x^m)^n = x^{m\cdot n} \qquad (x\cdot y)^n = x^n \cdot y^n$$
$$x^{-1} = \frac{1}{x} \qquad x^{-n} = \frac{1}{x^n}$$
$$x^{\frac{1}{2}} = \sqrt{x} \qquad \sqrt{x}\cdot\sqrt{x} = x$$
$$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$
$$x^{\frac{m}{n}} = \sqrt[n]{x^m} \qquad x^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{x^m}}$$

Logarithms

$$\log_a a^x = x \qquad a^{\log_a b} = b$$

Let $a^x = b$, isolate x from the exponent: $\log_a a^x = x = \log_a b$
Let $\log_a x = b$, isolate x from the logarithm: $a^{\log_a x} = x = a^b$

Laws of logarithms

I: $\log_c a + \log_c b = \log_c(a\cdot b)$

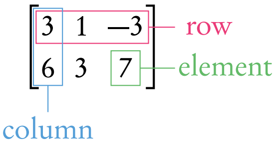
II: $\log_c a - \log_c b = \log_c\left(\frac{a}{b}\right)$

III: $n\log_c a = \log_c(a^n)$

1.7. Matrices 18

A matrix is a rectangular array of elements.

Below is a 2 × 3 matrix.



They can be multiplied together.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix} = \begin{bmatrix} ae+bh & af+bi & ag+bj \\ ce+dh & cf+di & cg+dj \end{bmatrix}$$

Rules of matrix multiplication.

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$
$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$\mathbf{A}\times\mathbf{B} \neq \mathbf{B}\times\mathbf{A}$$

1.1 Sequences and series

1.1.1 Arithmetic sequences

Arithmetic sequence the next term is the previous number + the common difference (d).

To find the common difference d, subtract two consecutive terms of an arithmetic sequence from the term that follows it, i.e. $d = u_{(n+1)} - u_n$.

Use the following equations to calculate the n^{th} term or the sum of n terms.

DB 1.2

$$u_n = u_1 + (n-1)d \qquad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

$$\text{with} \quad u_1 = a = 1^{\text{st}} \text{ term} \qquad d = \text{common difference}$$

Often the IB requires you to first find the 1st term and/or common difference.

Finding the first term u_1 and the common difference d from other terms

In an arithmetic sequence $u_{10} = 37$ and $u_{22} = 1$. Find the common difference and the first term.

1.	Put numbers into n^{th} term formula.	$37 = u_1 + 9d$ $1 = u_1 + 21d$
2.	Equate formulas to find d (using substitution method to solve simultaneous equations).	$21d - 1 = 9d - 37$ $12d = -36$ $d = -3$
3.	Use d to find u_1.	$1 - 21 \cdot (-3) = u_1$ $u_1 = 64$

1.1.2 Geometric sequence

Geometric sequence the next term is the previous number multiplied by the common ratio (r).

To find the common ratio, divide any term of an arithmetic sequence by the term that precedes it, i.e. $\frac{\text{second term } (u_2)}{\text{first term } (u_1)}$

Use the following equations to calculate the n^{th} term, the sum of n terms or the sum to infinity when $-1 < r < 1$.

DB 1.3 & 1.8

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1} \qquad S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)} \qquad S_\infty = \text{sum to infinity} = \frac{u_1}{1-r}$$

again with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad r = \text{common ratio}$$

Similar to questions on Arithmetic sequences, you are often required to find the 1st term and/or common ratio first.

1.2 Finance

1.2.1 Simple interest

Simple interest is given by the following formula.

$$I = P \times r \times n$$

where I = amount of interest, P = principal amount, r = interest rate per annum (as decimal), n = no. years

This works like an arithmetic sequence; for each year that interest is counted over a principle sum, a fixed interest rate is charged. This yearly interest is simply a percentage of the principal sum.

Example.

$1500 is invested at 5.25% simple interest per year. How much interest would be earned after 6 years?

$$\begin{aligned} I &= 1500 \times 0.0525 \times 6 \\ &= \$472.50 \end{aligned}$$

1.2.2 Compound interest

Compound interest refers to interest being added to an investment or principle sum every compounding period (instead of e.g. being paid out each time). This means that every time you calculate interest, you do so on a principle sum + previous interest. This works like a geometric sequence.

DB 1.4

$$\mathrm{FV} = \mathrm{PV} \times \left(1 + \frac{r}{100k}\right)^{kn}$$

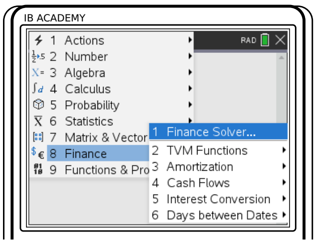
where FV = future value,
PV = present value,
n = no. years,
k = no. compounding periods per year,
r = % interest rate.

Besides using the equation in your data booklet, you can also use the TVM Solver (“Time Value of Money”) on your GDC to solve compound interest questions.

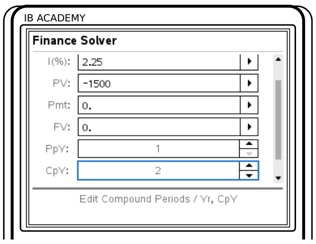
Abbreviation	Stands for
TVM	Time Value of Money
N	Number of payments
I%	percentage Interest rate
PV	Present Value - should be negative
PMT	PayMenT
FV	Future Value
P/Y	Payments per Year
C/Y	Compounding periods per Year

Solving questions about compound interest

$1500 is invested at 5.25% per annum. The interest is compounded twice per year. How much will it be worth after 6 years?

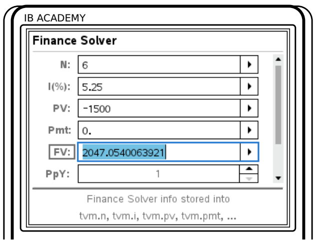


Press menu, choose
`8: Finance`
`1: Finance Solver`



I(%): 2.25
PV: -1500
Pmt: 0.
FV: 0.
PpY: 1
CpY: 2

Enter all known values
For this example:
N=12 (payments)
I=5.25 (interest rate)
PV=-1500 (present value)
negative because investment represents cash outflow;
PMT=0 FV=0 (future value)
P/Y=1 (payment/yr)
C/Y=2 (compound/yr)



N: 6
I(%): 5.25
PV: -1500
Pmt: 0.
FV: 2047.0540063921
PpY: 1

Highlight cell of asked value, in this case FV, press enter

So FV = $2793.62

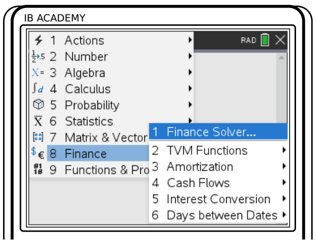
1.2.3 Annuities and amortization

Annuity a series of equal cash flows over equal periods in time

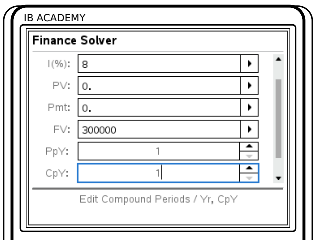
Amortization the process of spreading out a payment into a series of equal instalments over time

Finding the payment amount of an annuity

Jerome would like to save $300,000 to buy an apartment in 15 years' time. If he can invest at an 8% interest rate per year, how much money would he need to invest at the end of each year to reach his goal?



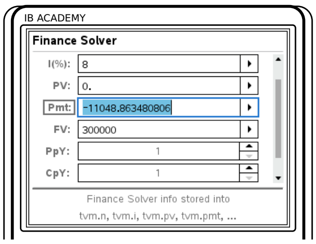
Press (menu), choose
`8: Finance`
`1: Finance Solver`



I(%): 8
PV: 0.
Pmt: 0.
FV: 300000
PpY: 1
CpY: 1

Edit Compound Periods / Yr, CpY

Enter all known values
For this example:
N=15 (payments)
I=8 (interest rate)
PV=0 (present value)
PMT=0
FV=300000 (future value)
P/Y=1 (payment/yr)
C/Y=1 (compound/yr)



I(%): 8
PV: 0.
Pmt: -11048.863480806
FV: 300000
PpY: 1
CpY: 1

Finance Solver info stored into
tvm.n, tvm.i, tvm.pv, tvm.pmt, ...

Highlight cell of asked value, in this case PMT, press (enter)

PMT $= -\$11,048.86$, so he would need to invest this amount each year

1.3 Estimations

Error by how much an estimate differs from the actual value

This can be found by calculating the approximate value − exact value

$$V_A - V_E$$

Percentage error $\left|\dfrac{\text{approximate value} - \text{exact value}}{\text{exact value}}\right| \times 100$

$$\left|\frac{V_A - V_E}{V_E}\right| \times 100$$

DB 1.6

Example.

John estimates a 119.423 cm piece of plywood to be 100 cm. What is the error?

$$\begin{aligned}\text{Error} &= V_A - V_E\\ &= 100 - 119.423\\ &= -19.423 \approx -19.4\end{aligned}$$

What is the percentage error?

$$\begin{aligned}\text{Percentage error} &= \left|\frac{100 - 119.423}{119.423}\right| \times 100\\ &= |-0.1626| \times 100\\ &= 0.1626 \times 100 \approx 16.3\%\end{aligned}$$

1.4 Using technology to solve linear equations and polynomial equations

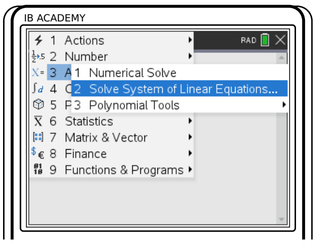
In algebraic problems where you have two unknown variables, for example x and y, and two equations, you can solve for x and y simultaneously.

The easiest way is to use the Simultaneous Equation Solver on your GDC.

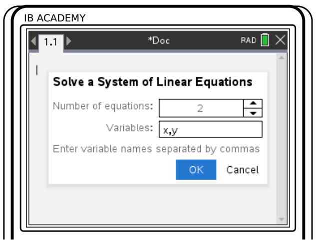
You will be expected to solve systems of up to 3 linear equations with this method.

Solving systems of linear equations

$y = 3x + 1$ (1) and $2y = x - 1$ (2); find the values of x and y



Press menu, choose
`3: Algebra`
`2: Solve System of`
`Linear Equations`



Solve a System of Linear Equations

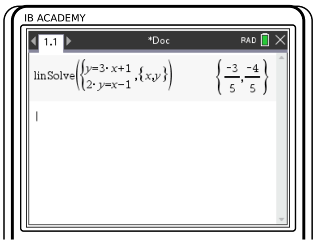
Number of equations: 2

Variables: x,y

Enter variable names separated by commas

`Number of equations = 2.`

Press OK



$\text{linSolve}\left(\begin{cases} y=3\cdot x+1 \\ 2\cdot y=x-1 \end{cases}, \{x,y\}\right) \quad \left\{\frac{-3}{5}, \frac{-4}{5}\right\}$

Enter the two equations

press enter

So $x = -\dfrac{3}{5}$ and $y = -\dfrac{4}{5}$

In case you prefer to solve a pair of simultaneous equations by hand, there are two methods you can use.

Example.

Elimination

Multiply an equation and then subtract it from the other in order to eliminate one of the unknowns.

$$3 \times (2) \Rightarrow \quad (3) \quad 6y = 3x - 3$$

$$(3) - (1) \Rightarrow \quad 6y - y = 3x - 3x - 3 - 1$$

$$5y = -4$$

$$y = -\frac{4}{5}$$

Put y in (1) or (2) and solve for x

$$-\frac{4}{5} = 3x + 1$$

$$3x = -\frac{9}{5}$$

$$x = -\frac{9}{15} = -\frac{3}{5}$$

Substitution

Rearrange and then substitute one in to another.
Substitute (1) into (2)

$$2(3x + 1) = x - 1$$

$$6x + 2 = x - 1$$

$$5x = -3$$

$$x = -\frac{3}{5}$$

Put x in (1) or (2) and solve for x

$$y = 3\left(-\frac{3}{5}\right) + 1$$

$$y = -\frac{4}{5}$$

1.5 Exponents and logarithms

1.5.1 Laws of exponents

Exponents always follow certain rules. If you are multiplying or dividing, use the following rules to determine what happens with the powers.

Example.

$$
\begin{aligned}
x^1 &= x & 6^1 &= 6 \\
x^0 &= 1 & 7^0 &= 1 \\
x^m \cdot x^n &= x^{m+n} & 4^5 \cdot 4^6 &= 4^{11} \\
\frac{x^m}{x^n} &= x^{m-n} & \frac{3^5}{3^4} &= 3^{5-4} = 3^1 = 3 \\
(x^m)^n &= x^{m \cdot n} & \left(10^5\right)^2 &= 10^{10} \\
(x \cdot y)^n &= x^n \cdot y^n & (2 \cdot 4)^3 &= 2^3 \cdot 4^3 \quad \text{and} \quad (3x)^4 = 3^4 x^4 \\
x^{-1} &= \frac{1}{x} & 5^{-1} &= \frac{1}{5} \quad \text{and} \quad \left(\frac{3}{4}\right)^{-1} = \frac{4}{3} \\
x^{-n} &= \frac{1}{x^n} & 3^{-5} &= \frac{1}{3^5} = \frac{1}{243}
\end{aligned}
$$

1.5.2 Fractional exponents

When doing mathematical operations (+, −, × or ÷) with fractions in the exponent you will need the following rules. These are often helpful when writing your answers in simplest terms.

Example.

$$
\begin{aligned}
x^{\frac{1}{2}} &= \sqrt{x} & 2^{\frac{1}{2}} &= \sqrt{2} \\
\sqrt{x} \cdot \sqrt{x} &= x & \sqrt{3} \cdot \sqrt{3} &= 3 \\
\sqrt{xy} &= \sqrt{x} \cdot \sqrt{y} & \sqrt{12} &= \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2 \cdot \sqrt{3} \\
x^{\frac{1}{n}} &= \sqrt[n]{x} & 5^{\frac{1}{3}} &= \sqrt[3]{5} \\
x^{\frac{m}{n}} &= \sqrt[n]{x^m} & 3^{-\frac{2}{5}} &= \frac{1}{\sqrt[5]{3^2}}
\end{aligned}
$$

1.5.3 Laws of logarithms

Logarithms are the inverse mathematical operation of exponents, like division is the inverse mathematical operation of multiplication. The logarithm is often used to find the variable in an exponent.

DB 1.5

$$a^x = b \quad \Leftrightarrow \quad x = \log_a b$$

Since $\log_a a^x = x$, so then $x = \log_a b$.
This formula shows that the variable x in the power of the exponent becomes the subject of your log equation, while the number a becomes the base of your logarithm.

Logarithms with bases of 10 and e have special notations in which their base is not explicitly noted.

$$\log_{10} x = \log x \qquad\qquad \log_{\mathrm{e}} x = \ln x$$

Remember that e is just the irrational number $2.71828\ldots$, so the same laws apply to $\ln$ as to other logarithms.

Below are the rules that you will need to use when performing calculations with logarithms and when simplifying them. The sets of equations on the left and right are the same; on the right we show the notation that the formula booklet uses while the equations on the left are easier to understand.

Laws of logarithms

DB 1.7

I:	$\log A + \log B = \log(A \cdot B)$	$\log_a(xy) = \log_a x + \log_a y$
II:	$\log A - \log B = \log\left(\frac{A}{B}\right)$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
III:	$n \log A = \log(A^n)$	$\log_a(x^m) = m \log_a x$

Next to these rules, there are a few handy things to keep in mind when working with logarithms.

$\log_a 0 = x$ is always undefined (because $a^x \neq 0$)
$x = \log_a a = 1$, which also means that $\ln \mathrm{e} = 1$
$\mathrm{e}^{\ln a} = a$

Solve for x in the exponent using logarithms

Solve $2^x = 13$

1.	Take the $\log$ on both sides.	$\log 2^x = \log 13$
2.	Use rule III to take x outside.	$x \log 2 = \log 13$
3.	Solve.	$x = \dfrac{\log 13}{\log 2}$

1.6 Complex numbers

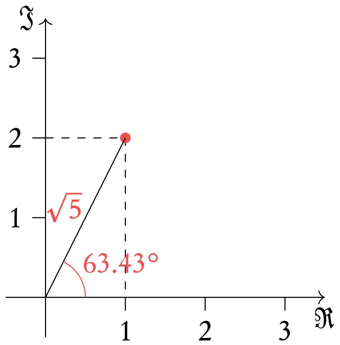
A complex number is defined as $z = a + b\mathrm{i}$. Where $a, b \in \mathbb{R}$, a is the real part ($\Re$) and b is the imaginary part ($\Im$).

$$\mathrm{i} = \sqrt{-1}$$
$$\mathrm{i}^2 = -1$$

$z = a + b\mathrm{i}$ is the Cartesian form. $z = r(\cos\vartheta + \mathrm{i}\sin\vartheta)$ is the polar form where r is the modulus and ϑ is the argument also sometimes stated as $z = r \operatorname{cis}\vartheta$.

Modulus r the absolute distance from the origin to the point.

Argument ϑ the angle between the x-axis and the line connecting the origin and the point.



Instead of working in (x, y) coordinates, polar coordinates use the distance from the origin to the point (r, modulus) and the angle between the x-axis and the modulus (argument).

$2 = 1 + 2\mathrm{i} \Rightarrow r = \sqrt{1^2 + 2^2} = \sqrt{5}$

and $\vartheta = \arctan(2) = 63.43°$

and $\sqrt{5} \times \sin(63.43) = 2$,

$\sqrt{5} \times \cos(63.43) = 1$

Complex numbers in Argand diagrams, like the axes above, can also be written as vectors. The top number in the vector refers to the real part of the complex number, and the bottom number to the imaginary part.

$$z = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The conjugate of a complex number $\overline{z}$ or z^*, is defined as $z = a + b\mathrm{i} \Rightarrow \overline{z} = a - b\mathrm{i}$

1.6.1 Complex numbers in the Cartesian form

Adding and subtracting complex numbers in Cartesian form is fairly straight forward. Add real and imaginary parts to each other:

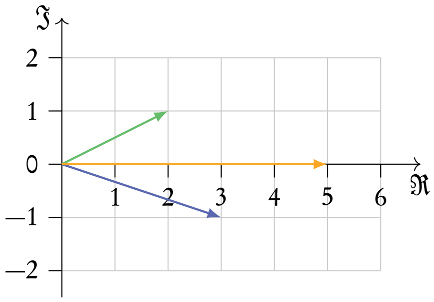
$$(2+3\mathrm{i})+(4+9\mathrm{i})=2+4+3\mathrm{i}+9\mathrm{i}=6+12\mathrm{i}$$

Multiplying complex numbers is like multiplying two parentheses:

$$\begin{aligned}(3-2\mathrm{i})(4+3\mathrm{i}) &= 3\times 4-2\mathrm{i}\times 4+3\times 3\mathrm{i}-2\mathrm{i}\times 3\mathrm{i}\\ &= 12-8\mathrm{i}+9\mathrm{i}-6\mathrm{i}^2\\ &= 12+6+\mathrm{i}\\ &= 18+\mathrm{i}\end{aligned}$$

We can also think about adding complex numbers in the form of vectors.

$$\begin{bmatrix}3\\-1\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$$



Division, however, is slightly more complex. Conjugates play a big role here, since a complex number multiplied by its conjugate is always equal to a real number.

Rewriting of a fraction with complex numbers

Rewrite $\dfrac{2+6\mathrm{i}}{1-2\mathrm{i}}$ in $a+b\mathrm{i}$ form.

1.	Convert the denominator into a real number by multiplying it with its conjugate.	$\dfrac{(2+6\mathrm{i})(1+2\mathrm{i})}{(1-2\mathrm{i})(1+2\mathrm{i})}$
2.	Expand the brackets and simplify, remember that $\mathrm{i}^2=-1$.	$\dfrac{2+6\mathrm{i}+4\mathrm{i}+12\mathrm{i}^2}{1-2\mathrm{i}+2\mathrm{i}-4\mathrm{i}^2}=\dfrac{-10+10\mathrm{i}}{5}$
3.	Write in $a+b\mathrm{i}$ form.	$-2+2\mathrm{i}$

1.6.2 Complex numbers in the polar form

Polar form allows us to do some operations quicker and more efficient, such as multiplication and division of complex numbers. The formulas can be shown for the following two complex numbers $z_1 = r_1 \operatorname{cis}(\vartheta_1)$ and $z_2 = r_2 \operatorname{cis}(\vartheta_2)$. Note: $\operatorname{cis} x = \cos x + \mathrm{i} \sin x$.

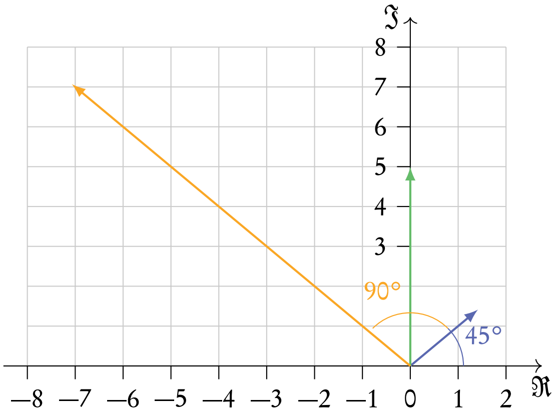
Example.

Multiplication: $z_1 \times z_2 = r_1 \times r_2 \operatorname{cis}(\vartheta_1 + \vartheta_2)$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis}(\vartheta_1 - \vartheta_2)$

Multiplying two complex numbers together can be visualised as a rotation and a stretch in an Argand diagram.

$2 \operatorname{cis}(45°) \times 5 \operatorname{cis}(90°) = 10 \operatorname{cis}(135°)$



Complex numbers can be useful in adding together two sinusoidal functions with the same frequency. A cosine function can be thought of as the real part of a complex number, and a sine function as the imaginary part.

$$\cos(3x) = \mathfrak{R}\left(\cos(3x) + \mathrm{i}\sin(3x)\right) = \mathfrak{R}\left(\mathrm{e}^{3x\mathrm{i}}\right)$$
$$\sin(3x) = \mathfrak{I}\left(\cos(3x) + \mathrm{i}\sin(3x)\right) = \mathfrak{I}\left(\mathrm{e}^{3x\mathrm{i}}\right)$$

Add cosine functions together

Add the following cosine functions together:

$$f_1 = 10\cos(40t)$$
$$f_2 = 20\cos(40t+10)$$

1. Write the functions as real/imaginary parts of a complex number.

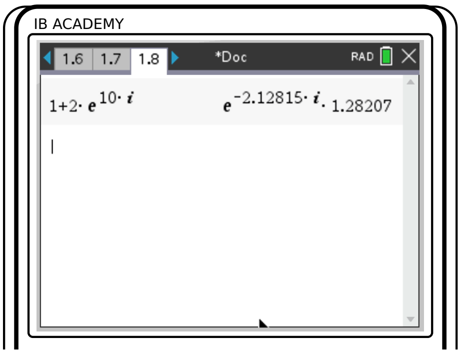
$$f_1 = \Re\left(10e^{40ti}\right)$$
$$f_2 = \Re\left(20e^{(40t+10)i}\right) = \Re\left(20e^{40ti}e^{10i}\right)$$

2. Add these new expressions together and take out a factor.

$$\begin{aligned} f_1 + f_2 &= \Re\left(10e^{40ti}\right) + \Re\left(20e^{40ti}e^{10i}\right) \\ &= \Re\left(10e^{40ti} + 20e^{40ti}e^{10i}\right) \\ &= \Re\left(10e^{40ti}\left(1+2e^{10i}\right)\right) \end{aligned}$$

3. Simplify the complex number with no variables (such as t) using your calculator.

Your calculator should express it as a single complex number. You may have to change the document settings to express it in polar form.



$1+2\cdot e^{10\cdot i}$ $\quad e^{-2.12815\cdot i}\cdot 1.28207$

4. Use the simplified complex number to simplify the entire expression to one complex number.

$$\begin{aligned} f_1 + f_2 &= \Re\left(10e^{40ti}\left(1.282e^{-2.128i}\right)\right) \\ &= \Re\left(12.82e^{-2.128i+40ti}\right) \\ &= \Re\left(12.82e^{i(-2.128+40t)}\right) \end{aligned}$$

5. Find the real part of this complex number.

$$f_1 + f_2 = 12.82\cos(40t - 2.128)$$

Euler's and De Moivre's theorem

These two theorems state the relationship between the trigonometric functions and the complex exponential function. This allows us to convert between Cartesian and Polar forms.

Euler's Theorem $e^{ix} = \cos x + i \sin x$

De Moivre's theorem $z^n = \left(r(\cos x + i \sin x)\right)^n = r^n \left(\cos(nx) + i \sin(nx)\right)$

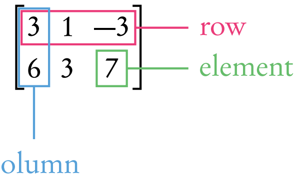
De Moivre's theorem can be derived from Euler's through the exponential law for integer powers. $(e^{ix})^n = e^{ixn} = z^n$

1.7 Matrices

1.7.1 Introduction

A matrix is a rectangular array of elements, usually numbers. They consist of rows (horizontal) and columns (vertical) of elements. An m by n, or $m \times n$, matrix is one which has m rows and n columns. Matrices can describe transformations, and all vectors are a type of matrix.

Below is a 2×3 matrix.



Matrices can be added together or subtracted from one another as long as they have the same dimensions (same number of rows and columns). This is done by adding or subtracting the corresponding elements from each matrix.

$$\begin{bmatrix} 3 & 6 & -5 \\ 4 & 2 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 7 & 4 \\ -3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -9 \\ 7 & 3 & 5 \end{bmatrix}$$

Matrices can also be multiplied by a scalar by multiplying each element by the scalar.

$$7 \begin{bmatrix} -2 & 20 & 16 \\ 3 & -7 & 8 \end{bmatrix} = \begin{bmatrix} -14 & 140 & 112 \\ 21 & -49 & 56 \end{bmatrix}$$

1.7.2 Matrix multiplication

Matrices can be multiplied by matrices to give another matrix. When multiplying matrices together the number of columns of the first matrix must be equal to the number of rows of the second matrix. For example we can multiply a 3×4 matrix by a 4×5 matrix, but not by a 6×3 matrix. When multiplying an $m \times n$ matrix by an $n \times o$ matrix the resulting matrix will have the dimensions $m \times o$.

To find the elements in the resulting matrix the dot product can be used. The dot product will also be used in the vectors topic. To take the dot product of two sets of numbers the first terms in each set are multiplied together, and then the second terms, and the third terms and so on. These are all added together to give the dot product.

Example.

Calculate the dot product between these two vectors.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 4 + 10 + 18 = 32$$

To multiply two matrices together the dot product is used. To calculate the element in the first row and first column of the resultant matrix, take the dot product between the first row of the first matrix and the first column of the second matrix. Only rows from the first matrix are used and only columns from the second matrix. You can think of it as multiplying along the first matrix and down the second matrix.

$$\overset{\longrightarrow}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}\Big\downarrow = \begin{bmatrix} ae + bh & af + bi & ag + bj \\ ce + dh & cf + di & cg + dj \end{bmatrix}$$

Matrix multiplication

Multiply the following matrices together.

$$\begin{bmatrix}3 & 2\\4 & 6\end{bmatrix}\begin{bmatrix}3 & 4 & -1\\4 & 0 & 3\end{bmatrix}$$

1.	Check the dimensions of the matrices.	The first matrix has two columns, and the second matrix has two rows, so we can multiply them together.
2.	Find the dimensions of the resultant matrix.	We have a 2×2 matrix multiplying a 2×3 matrix, this gives a 2×3 matrix. $\begin{bmatrix}3 & 2\\4 & 6\end{bmatrix}\begin{bmatrix}3 & 4 & -1\\4 & 0 & 3\end{bmatrix}=\begin{bmatrix}? & ? & ?\\? & ? & ?\end{bmatrix}$
3.	Multiply the first row of the first matrix by the first column of the second matrix.	$\begin{bmatrix}3 & 2\\4 & 6\end{bmatrix}\begin{bmatrix}3 & 4 & -1\\4 & 0 & 3\end{bmatrix}=\begin{bmatrix}3\times 3+2\times 4 & ? & ?\\? & ? & ?\end{bmatrix}=\begin{bmatrix}17 & ? & ?\\? & ? & ?\end{bmatrix}$
4.	Repeat this for the first row and second column.	$\begin{bmatrix}3 & 2\\4 & 6\end{bmatrix}\begin{bmatrix}3 & 4 & -1\\4 & 0 & 3\end{bmatrix}=\begin{bmatrix}17 & 3\times 4+2\times 0 & ?\\? & ? & ?\end{bmatrix}=\begin{bmatrix}17 & 12 & ?\\? & ? & ?\end{bmatrix}$
5.	Continue to repeat this for all elements in the resultant matrix.	$\begin{bmatrix}3 & 2\\4 & 6\end{bmatrix}\begin{bmatrix}3 & 4 & -1\\4 & 0 & 3\end{bmatrix}=$ $=\begin{bmatrix}17 & 12 & 3\times -1+2\times 3\\4\times 3+6\times 4 & 4\times 4+6\times 0 & 4\times -1+6\times 3\end{bmatrix}=$ $=\begin{bmatrix}17 & 12 & 3\\36 & 16 & 14\end{bmatrix}$

Matrix multiplication

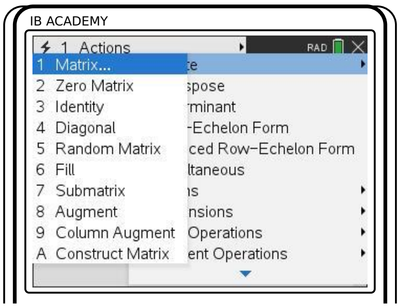
Let $A = \begin{bmatrix} 9 & 11 \\ -2 & 7 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 12 & 4 & 24 \\ 1 & -7 & 8 \end{bmatrix}$. Find $\mathbf{AB}$.

1. Check dimensions of the matrices.

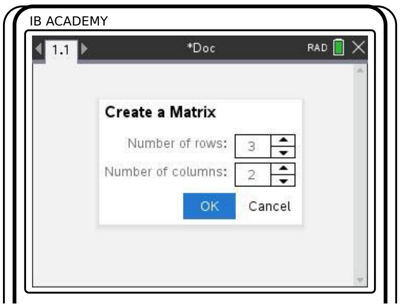
The first matrix has two columns and the second matrix has two rows, so we can multiply them together.

2. Solve using GDC.

Input the matrices into your calculator.



Press menu, choose
`7: Matrix & vector`
`1: Create`
`1: Matrix`



Create a Matrix
Number of rows: 3
Number of columns: 2

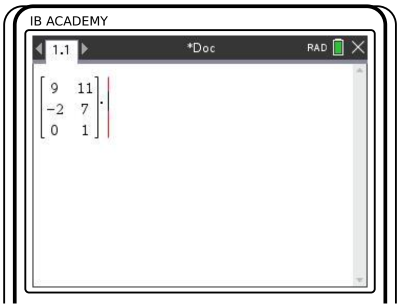
Enter dimensions of matrix

Press OK

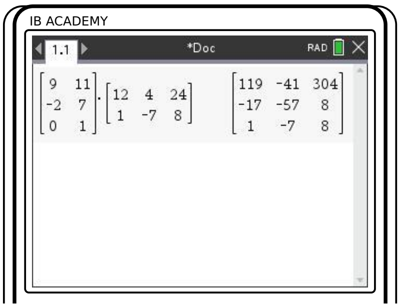
Enter all matrix values

3. Solve using GDC.

Multiply matrices together.



$\begin{bmatrix} 9 & 11 \\ -2 & 7 \\ 0 & 1 \end{bmatrix} \cdot$



$\begin{bmatrix} 9 & 11 \\ -2 & 7 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 12 & 4 & 24 \\ 1 & -7 & 8 \end{bmatrix} \quad \begin{bmatrix} 119 & -41 & 304 \\ -17 & -57 & 8 \\ 1 & -7 & 8 \end{bmatrix}$

Press ×

Enter second matrix repeating above steps

Press enter

$$\mathbf{AB} = \begin{bmatrix} 119 & -41 & 304 \\ -17 & -57 & 8 \\ 1 & -7 & 8 \end{bmatrix}$$

Properties

Matrix multiplication has some properties which are listed below, where **A**, **B**, and **C** are all matrices.

Multiplying matrices is different to multiplying numbers, the matrices cannot switch places and give the same resultant matrix.

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$$
$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=\mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$
$$\mathbf{A}\times\mathbf{B}\neq\mathbf{B}\times\mathbf{A}$$

Example.

$$\mathbf{A}=\begin{bmatrix}3 & 6\\ 4 & -1\end{bmatrix} \qquad \mathbf{B}=\begin{bmatrix}2 & 3\\ -2 & 8\end{bmatrix}$$
$$\mathbf{A}\times\mathbf{B}=\begin{bmatrix}-6 & 57\\ 10 & 4\end{bmatrix} \qquad \mathbf{B}\times\mathbf{A}=\begin{bmatrix}18 & 9\\ 26 & -20\end{bmatrix}$$

1.7.3 Definitions

A zero matrix is one with zeros in every place. These can be of any size and are written as $0_{n\times m}$ where n and m are the dimensions of the matrix.

$$0_{2\times2}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \qquad 0_{5\times3}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

An identity matrix is a square matrix (it has the same number of rows as columns) with zeros in every entry apart from the main diagonal, which has 1 as each entry. It is written as $\mathbf{I}_n$ where n is the size of the matrix. If we multiply a matrix by an identity matrix (of the same dimensions) our matrix will remain the same. It is the equivalent of multiplying a number by the number 1.

$$\mathbf{I}_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \mathbf{I}_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad \mathbf{I}_4=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

The determinant of a matrix, **A**, is written as $\det\mathbf{A}$ or $|\mathbf{A}|$ and is calculated using the following formula. The determinant tells us something about the transformations which the matrix represents.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\det \mathbf{A} = |\mathbf{A}| = ad - bc$$

DB 1.14

The inverse of a matrix **A** is written as $\mathbf{A}^{-1}$. When a matrix is multiplied by its inverse it gives the identity matrix. This is how the inverse matrix is defined. The following formula shows how to calculate the inverse of a matrix.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

DB 1.14

The determinant and inverse of matrices larger than 2×2 can be calculated using your calculator. The IB only expects you to calculate inverses and determinants of 2×2 matrices by hand.

Inverse of matrix

Find the inverse of matrix **A**.

1. First we calculate the determinant.

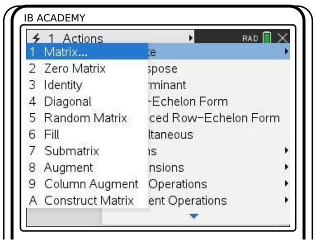
$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & 9 \end{bmatrix}$$
$$\det \mathbf{A} = |\mathbf{A}| = 2 \times 9 - 4 \times -3 = 18 + 12 = 30$$

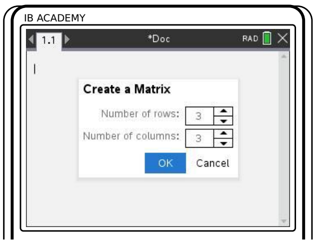
2. Filling this into the formula for our inverse matrix.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 9 & 3 \\ -4 & 2 \end{bmatrix}$$

Inverse matrices

Let $\mathbf{A} = \begin{bmatrix} 1 & 4 & -3 \\ 3 & 0 & 1 \\ 7 & 2 & 2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

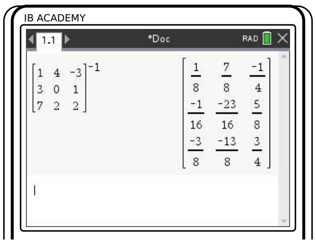




Create a Matrix

Number of rows: 3

Number of columns: 3



$$\begin{bmatrix} 1 & 4 & -3 \\ 3 & 0 & 1 \\ 7 & 2 & 2 \end{bmatrix}^{-1} \quad \begin{bmatrix} \frac{1}{8} & \frac{7}{8} & \frac{-1}{4} \\ \frac{-1}{16} & \frac{-23}{16} & \frac{5}{8} \\ \frac{-3}{8} & \frac{-13}{8} & \frac{3}{4} \end{bmatrix}$$

Press menu, choose
7: Matrix & vector
1: Create
1: Matrix

Enter dimensions of matrix
Press OK
Enter all matrix values

Raise the matrix to the power of -1
Press enter

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{7}{8} & \frac{-1}{4} \\ \frac{-1}{16} & \frac{-23}{16} & \frac{5}{8} \\ \frac{-3}{8} & \frac{-13}{8} & \frac{3}{4} \end{bmatrix}$$

1.7.4 Solving equations

The inverse of matrices are useful in solving equations of the form $\mathbf{Ax} = \mathbf{b}$. This is just another way to write simultaneous equations. $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{x}$ is an $n \times 1$ matrix, or vector, and $\mathbf{b}$ is an $m \times 1$ matrix, or vector.

An example is given below.

$$\begin{bmatrix} 2 & 6 & 7 \\ 5 & -3 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Multiplying this out gives a more familiar looking equation.

$$\begin{gathered} \begin{bmatrix} 2x_1 + 6x_2 + 7x_3 \\ 5x_1 - 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ 2x_1 + 6x_2 + 7x_3 = 3 \\ 5x_1 - 3x_2 - 8x_3 = 5 \end{gathered}$$

In order to solve these equations we must use the inverse of matrix $\mathbf{A}$ to rearrange them. The goal is to end up with only $\mathbf{x}$ on one side and only numbers on the other side. Since we are unable to divide by a matrix, we multiply both sides by the inverse to get rid of matrix $\mathbf{A}$ on the left side of the equation. Multiplying $\mathbf{A}^{-1}$ by $\mathbf{A}$ gives the identity matrix, $\mathbf{I}$, which is equivalent to the number 1.

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{A}^{-1}\mathbf{Ax} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{Ix} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

Now we have an equation which has $\mathbf{x}$ on one side and numbers on the other side.

Solving an equation using matrices

Solve the following equation.

$$\begin{bmatrix} 4 & -8 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

1. Calculate the inverse of the matrix with known values.

$$\det\left(\begin{bmatrix} 4 & -8 \\ 2 & 3 \end{bmatrix}\right) = 12 - (-16) = 28$$

$$\mathbf{A}^{-1} = \frac{1}{28}\begin{bmatrix} 3 & 8 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{28} & \frac{2}{7} \\ -\frac{1}{14} & \frac{1}{7} \end{bmatrix}$$

2. Multiply both sides of the equation by the inverse to cancel out the matrices on the left hand side.

$$\begin{bmatrix} \frac{3}{28} & \frac{2}{7} \\ -\frac{1}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 4 & -8 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{28} & \frac{2}{7} \\ -\frac{1}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{28} & \frac{2}{7} \\ -\frac{1}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

3. Find the unknown values by the multiplying the matrices on the right hand side.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{71}{28} \\ \frac{9}{14} \end{bmatrix}$$

1.7.5 Eigenvectors and eigenvalues

An eigenvector is a vector which, when multiplied by a matrix gives the same vector multiplied by a factor known as an eigenvalue. Eigenvalues are scalars, they are just a number. Eigenvectors are often written as **v** and their corresponding eigenvalues as λ. Eigenvectors are always specific to a matrix and always have a corresponding eigenvalue. The IB only expects you to calculate eigenvalue and eigenvectors of 2×2 matrices.

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

For example the following vector, **v**, is an eigenvector with eigenvalue $\lambda = -1$, of the matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can multiply this matrix by its eigenvector, and the matrix it gives us is the eigenvector multiplied by its eigenvalue.

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 1 \times -1 \\ -2 \times 1 + -3 \times -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

To find the eigenvalues of a matrix we can use something know as the characteristic equation. The previous equation can be rewritten.

$$\mathbf{A}\mathbf{v} - \lambda\mathbf{v} = 0$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$$

In order for this equation to have solutions the following must be true. The matrix **I** refers to the identity matrix which is the same size as the matrix **A**.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This equation is used to calculate the eigenvalues. Once the eigenvalues are known the eigenvectors can be calculated.

Eigenvectors and eigenvalues of a matrix

Find the eigenvalues and eigenvectors of the following matrix.

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$$

1.	Find $\mathbf{I}$.	In this case $\mathbf{I}$ will be a 2×2 matrix, the same size as our matrix $\mathbf{A}$. $\mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
2.	Enter the matrices into the characteristic equation and simplify.	Characteristic equation $\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=0$. $\det\left(\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=0$ $\det\left(\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)=0$ $\det\left(\begin{bmatrix}-\lambda & 1\\ -2 & -3-\lambda\end{bmatrix}\right)=0$
3.	Calculate the determinate using the formula.	$\left(-\lambda\times\left(-3-\lambda\right)\right)-(1\times -2)=0$ $3\lambda+\lambda^2+2=0$
4.	Solve the quadratic equation to find λ, the eigenvalues, using the quadratic equation.	This is known as the characteristic polynomial – the polynomial whose roots are the eigenvalues of a matrix. $\lambda=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm 1}{2}$ $\lambda_1=-1$ $\lambda_2=-2$

5.	Use the eigenvalues to find their corresponding eigenvectors.	We will begin with the first eigenvalue $\lambda_1 = -1$. Substitute it, along with the matrices, into the formula $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{v}_1 = 0$. The eigenvector is written out in terms of its components $v_{1,1}$ and $v_{1,2}$. $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{v}_1 = 0$ $\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\right)\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$ $\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$ $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$ $\begin{bmatrix} v_{1,1} + v_{1,2} \\ -2v_{1,1} - 2v_{1,2} \end{bmatrix} = 0$
6.	Simplify the equations given. This gives us two equations which link together the components of the eigenvector. Find the equation which they both reduce to.	$v_{1,1} + v_{1,2} = 0$ $-2v_{1,1} - 2v_{1,2} = 0$ Both of these reduce to the equation $v_{1,1} = -v_{1,2}$.
7.	Every eigenvalue will have multiple eigenvectors associated with it, so we can not solve for a specific vector. Instead, give a value to one component of the vector and work out the other component using the equations above.	Set $v_{1,1} = 1$. Then work out $v_{1,2}$. $v_{1,2} = -v_{1,1} = -1$
8.	Fill in the values of the vector's components into the vector.	This is your first eigenvector, $\mathbf{v}_1$ and its eigenvalue is $\lambda_1 = -1$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
9.	Repeat steps **5.** to **8.** for the second eigenvalue.	

1.7.6 Diagonalisation of matrices

Eigenvalues and eigenvectors allow us to easily raise any square matrix to high powers. A matrix, **M**, can be written using the following equation, where the matrices **D** and **P** make use of the eigenvalues and eigenvectors of the matrix.

DB 1.15

$$\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$$

The IB only expects you to work with this equation for 2 × 2 matrices. The eigenvalues of a matrix, **M**, can be used to create the diagonalised matrix, **D**. This matrix has zeros everywhere apart from on the main diagonal, on which are the eigenvalues of the matrix **M**.

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The other matrix is known as an invertible matrix, **P**. This matrix has the eigenvectors of matrix **M** vertically as its entries.

$$\mathbf{P} = \begin{bmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{bmatrix}$$

The matrix **D** only has non-zero entries on its main diagonal, and we can apply the following formula. In order to raise it to a power we can raise each entry in the matrix to the same power. This works only because it is a diagonalised matrix.

$$\mathbf{D}^n = \begin{bmatrix} (\lambda_1)^n & 0 \\ 0 & (\lambda_2)^n \end{bmatrix}$$

Raise a matrix to a power

Raise the following matrix to the 5th power, by finding the matrices $\mathbf{D}$ and $\mathbf{P}$ and using the equation $\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$.

$$\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

1. Find the eigenvalues and eigenvectors of the matrix.

This was done in a previous example.

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2. Fill in the eigenvalues into the matrix.

$$\mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

3. Fill in the eigenvectors vertically into the matrix.

$$\mathbf{P} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

4. Find the inverse matrix of the matrix.

$\mathbf{P}^{-1}$ of matrix $\mathbf{P}$.

$$\det(\mathbf{P}) = (1 \times 2) - (-1 \times -1) = 2 - 1 = 1$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

5. Fill all these matrices into the equation.

$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

$$\mathbf{M} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

6. Raise to the required power using the equation.

$$\mathbf{M}^5 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^5 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{M}^5 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^5 & 0 \\ 0 & (-2)^5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{M}^5 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -32 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

7. Multiply out the matrices to give your answer in a single matrix.

Here, we begin with the first two matrices, then multiply the answer by the third matrix. Recall that matrix multiplication is associative – it does not matter in which order we multiply them.

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -32 \end{bmatrix} = \begin{bmatrix} -1 & 32 \\ 1 & -64 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 32 \\ 1 & -64 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 30 & 31 \\ -62 & -63 \end{bmatrix}$$

$$\mathbf{M}^5 = \begin{bmatrix} 30 & 31 \\ -62 & -63 \end{bmatrix}$$

FUNCTIONS

2

Table of contents & cheatsheet

Definitions

Function	a mathematical relationship where each input has a single output. It is often written as $f(x)$ where x is the input
Domain	all possible x values, the input. (the domain of investigation)
Range	possible y values, the output. (the range of outcomes)
Coordinates	uniquely determines the position of a point, given by (x, y)

2.1. Types of functions 34

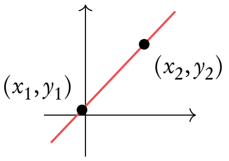
Linear functions $y = mx + c$

m is the *gradient*, c is the y *intercept*.

Midpoint: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Distance: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Gradient: $m = \frac{y_2-y_1}{x_2-x_1}$

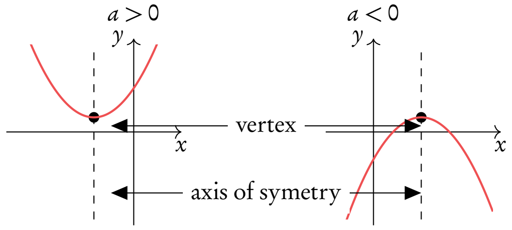


Parallel lines: $m_1 = m_2$ (same gradients)
Perpendicular lines: $m_1 m_2 = -1$

Quadratic functions $y = ax^2 + bx + c = 0$

Axis of symmetry: x-coordinate of the vertex: $x = \frac{-b}{2a}$

Factorized form: $y = (x+p)(x+q)$



If $a = 1$ use the factorization method $(x+p)\cdot(x+q)$

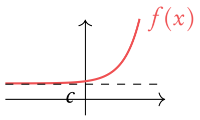
If $a \neq 1$ use the quadratic formula

When asked explicitly complete the square

Vertex form: $y = a(x-h)^2 + k$
Vertex: (h, k)

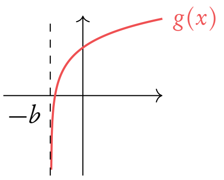
Exponential

$f(x) = a^x + c$



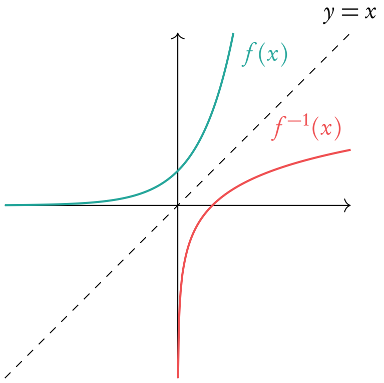
Logarithmic

$g(x) = \log_a(x+b)$



2.6. Rearranging functions 44

Inverse function, $f^{-1}(x)$ reflection of $f(x)$ in $y = x$.



Composite function, $(f \circ g)(x)$ is the combined function f of g of x.

When $f(x)$ and $g(x)$ are given, replace x in $f(x)$ by $g(x)$.

Transforming functions

Change to $f(x)$	Effect
$f(x)+a$	Move graph a units upwards
$f(x+a)$	Move graph a units to the left
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $\frac{1}{a}$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis

2.1 Basic concepts

2.1.1 Domain and range

A mathematical model transforms an input value into an output value. To describe a mathematical model (or function) you therefore need to know the possible x and y-values that it can have; these are called the domain and the range respectively.

Function a mathematical relationship where each input has a single output. It is often written as $y = f(x)$ where x is the input.

Domain all possible x-values that a function can have. You can also think of this as the 'input' into a mathematical model.

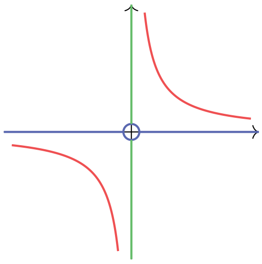
Range all possible y-values that a function can give you. You can also think of this as the 'output' of a mathematical model.

Coordinates uniquely determine the position of a point, given by (x, y).

Note that some questions will specify the domain (often even though the function as such could theoretically have many other x-value inputs). Make sure that your answers are within any given domain; for example, only sketch the graph for the x-values included in the domain if you are asked to draw it.

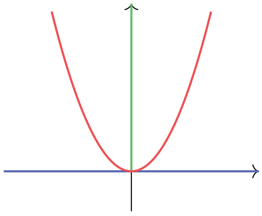
Example.

Find the domain and range for the function $y = \frac{1}{x}$



Domain: $x \neq 0$
(all real numbers except 0)
Range: $y \neq 0$
(all real numbers except 0)

Find the domain and range for the function $y = x^2$



Domain: $x \in \mathbb{R}$
(all real numbers)
Range: $y \in \mathbb{R}^+$
(all positive real numbers)

2.1.2 Sketching graphs

When sketching graphs, make sure your drawing is to scale and within the required domain and range.

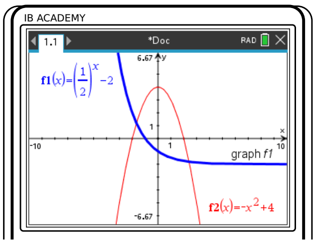
Even though the IB does not not expect your sketches to be completely precise, it is important that key features are in the right place. These include:

- x- and y-intercepts
- intersection points
- turning points
- axes of symmetry
- horizontal and vertical asymptotes

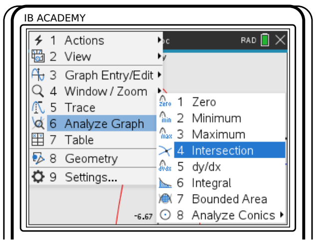
As well as sketching these features, you will need to know how to identify them on given graphs and on your GDC.

Finding a point of intersection

$f(x) = \left(\frac{1}{2}\right)^x - 2$ and $g(x) = -x^2 + 4$. Find the coordinates where $f(x) = g(x)$.



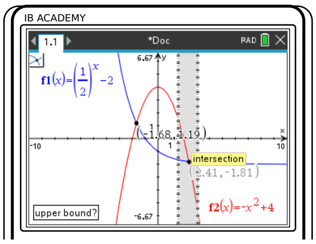
Plot both functions



Press menu

`6: Analyse Graph`

`4: Intersection`



Approach the intersection you are trying to find with the cursor and click once you near it. Repeat for any other intersections.

The intersection points are $(-1.68, 1.19)$ and $(2.41, -1.81)$. If you are sketching functions $f(x)$ and $g(x)$, be sure to mark and label these points on your sketch.

2.2 Linear models

Linear functions make straight line graphs. Two elements you need to know to describe a linear function are its slope/gradient (how steeply it is rising or decreasing) and its y-intercept (the y-value when the function crosses the y-axis, so when $x = 0$).

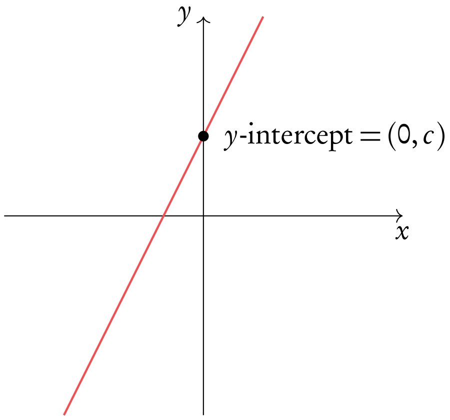
Straight line equation is usually written in the following form:

$$y = mx + c$$

with

$m =$ gradient (slope)

$c = y$-intercept



This is useful, because this way you can read the gradient (m) and y-intercept (c) directly from the equation (or formulate a straight line equation yourself, if you know the value of the gradient and y-intercept.)

You may also see a straight line equation written in two other forms:

$ax + by + d = 0$	general form
$y - y_1 = m(x - x_1)$	point-slope form

In these cases, it is best to rearrange the equation into the $y = mx + c$ form discussed above. You can do this by using the rules of algebra to make y the subject of the equation.

When you are not given the value of the gradient in a question, you can find it if you know two points that should lie on your straight line. The gradient (m) can be calculated by substituting your two known coordinates (x_1, y_1) and (x_2, y_2) into the following equation:

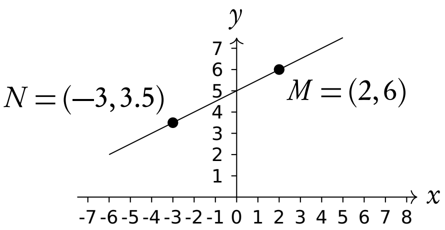
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Make sure you substitute the y and x-coordinates in the correct order!

DB 2.1

Finding the equation of a line and its x-intercept

Find the equation of the straight line that passes through points $M(2,6)$ and $N(-3,3.5)$. Then find the x-intercept.



1. Take two points on the graph and substitute the values into the formula.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{6-3.5}{2-(-3)}=0.5$$

2. Fill in one point to find c.

$$6=0.5(2)+c$$
$$6=1+c$$
$$c=5$$

3. Write down the equation $y=mx+c$ replacing m and c.

$$y=0.5x+5$$

4. To find the x-intercept, solve by isolating x.

$$0=0.5x+5$$
$$-5=0.5x$$
$$x=-10$$

x-intercept: (-10,0)

2.2.1 Intersection of lines

Finding a linear equation with a given gradient and intersection point

Line L_1 has a gradient of 5 and intersects line L_2 at point $A(1,0)$. Find the equation of L_1

1. Find slope.

Slope given, $m=5$

2. Fill in one point to find c.

L_1 passes through $(1,0)$

$$\Rightarrow 0=5(1)+c$$
$$c=-5$$
$$\Rightarrow y=5x-5$$

2.2.2 Parallel and perpendicular lines

When you know the equation of one straight line, you can use the value of its gradient, m, to find equations of other straight lines that are parallel or perpendicular to it.

Parallel lines	have the same slope	$m_1 = m_2$
Perpendicular lines	meet at a 90° angle	$m_2 = \frac{-1}{m_1}$

Finding a linear equation of a perpendicular line

Line L_1 has a gradient of 5 and intersects line L_2 at point $A(1,0)$. Line L_2 is perpendicular to L_1. Find the equation of L_2

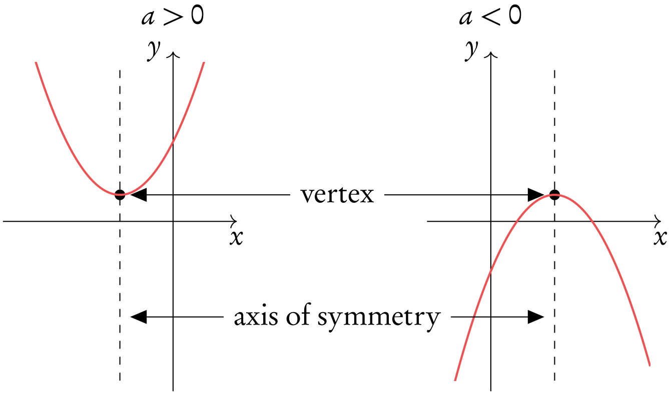
1.	Find slope.	L_2 is perpendicular to L_1 so $m = -\frac{1}{\text{gradient}}$ $\Rightarrow m = -\frac{1}{5}$
2.	Fill in one point to find c.	$0 = -\frac{1}{5}(1) + c$ $c = \frac{1}{5}$ $\Rightarrow y = -\frac{1}{5}x + \frac{1}{5}$

2.3 Quadratic models

A quadratic function consist of an equation with a term raised to the power of 2.

$$y = ax^2 + bx + c = 0$$

When plotted on a graph, a quadratic function makes an upward or downward facing parabolic shape. A parabola always has a vertex (the maximum or minimum point) and an axis of symmetry.



If you know the x and y coordinate of the vertex, the equation for the axis of symmetry will always be $x =$ [the x-coordinate of the vertex]. This also works the other way around; the equation of the axis of symmetry gives you the x-coordinate of the vertex.

The equation for the axis of symmetry can be found using the equation below where a, b and c are the corresponding numbers from your quadratic equation written in the form $y = ax^2 + bx + c$.

Axis of symmetry $x = \frac{-b}{2a} =$ x-coordinate of vertex

DB 2.5

Finding the vertex of a quadratic function

Given that $f(x) = x^2 - 2x - 15$, find the coordinates of the vertex of $f(x)$.

1.	Use axis of symmetry formula to find x-coordinate of the vertex.	$x = \frac{-b}{2a} = \frac{-(-2)}{2 \cdot 1}$ $\Rightarrow x = 1$
2.	Use $f(x)$ to find y-coordinate of the vertex.	$y = 1^2 - 2(1) - 15$ $\Rightarrow y = -16$ vertex: $(1, -16)$

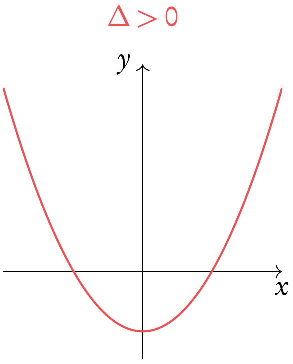
2.3.1 Solving quadratic functions

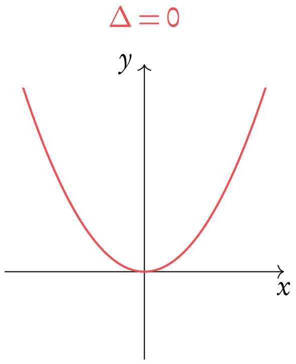
To ‘solve’ a quadratic function, you need to find its x-intercepts. You find these by setting your quadratic equation equal to 0. When $ax^2+bx+c=0$ you can solve for x to find the x-intercepts (or ‘roots’, or ‘solutions’ as they are also called interchangeably). Given that quadratic equations have the shape of a parabola, they can have up to two x-intercepts; as you can see when a quadratic equation is plotted, it often crosses the x-axis twice.

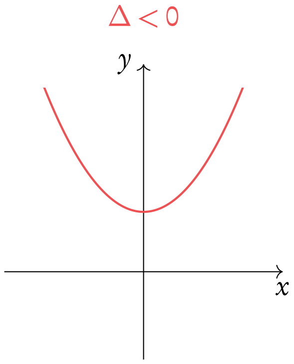
Quadratic functions can have either two real roots, one repeated real root or two complex roots. Something called the discriminant (Δ) can help us tell which roots the function has.

$$\Delta = b^2 - 4ac$$

If the discriminant is positive, the function has two real roots, if it is zero the function has one repeated root and if it is negative the function has two complex roots.







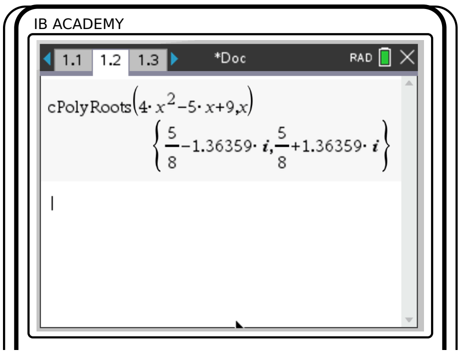
Solving for complex roots on GDC

Find the roots of the quadratic equation $4x^2 - 5x + 9$.

1. Check whether the roots are complex or real.

$$\Delta = b^2 - 4ac = (-5)^2 - 4 \times 4 \times 9 = -119$$
$$\Delta < 0$$

2. To find the complex roots of a polynomial, open a calculator page.



$\text{cPolyRoots}(4\cdot x^2-5\cdot x+9,x)$

$\left\{\frac{5}{8}-1.36359\cdot i,\frac{5}{8}+1.36359\cdot i\right\}$

Press menu

`3: Algebra`

`3: Polynomial Tools`

`1: Find Roots of Polynomial`

Select `Complex` roots and enter the coefficients of your equation.

There are several methods to find the x-intercepts. In your exam you will primarily use your GDC. Here we work through an example of factorisation.

Factorisation

Using factorisation to find the roots of a quadratic function

Factorise $f(x) = x^2 - 2x - 15$. Hence find the roots of $f(x) = 0$.

1. Rearrange $f(x) = 0$ into the form $f(x) = (x+p)(x+q)$.

Look for the pair of numbers that satisfy:

$$\begin{aligned} p + q &= -2 \\ pq &= 15 \end{aligned} \quad \Rightarrow \quad \begin{aligned} p &= -5 \\ q &= 3 \end{aligned}$$

so $f(x) = (x-5)(x+3)$

2. Solve $x + p = 0$ and $x + q = 0$ to find the roots (i.e. change the signs on p and q).

$$\begin{aligned} x - 5 &= 0 \\ x + 3 &= 0 \end{aligned} \quad \Rightarrow \quad \text{roots: } \begin{aligned} x &= 5 \\ x &= -3 \end{aligned}$$

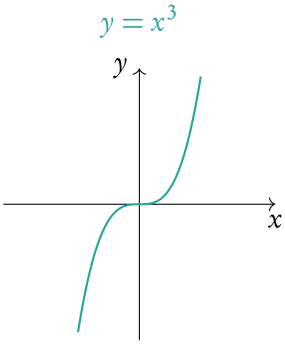
2.4 Polynomials

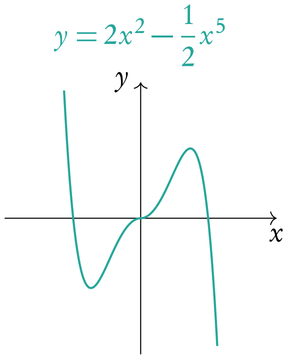
A polynomial function is made up of one or more summed terms, each of which is generally a variable (e.g. x) raised to a power and multiplied by a coefficient.

$$f(x) = ax^n$$

Example.

Use GDC to see what a given polynomial looks like.



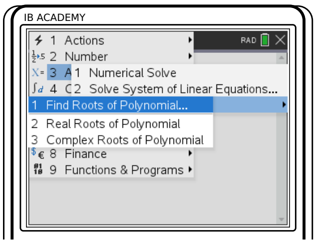


You can use the `Polyrootfinder` on your GDC to find roots of any polynomials.

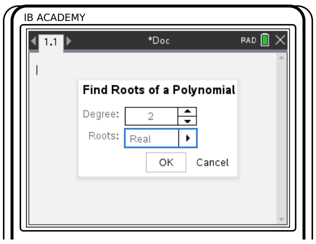
Solving polynomial equations on GDC

Solve $3x^2 - 4x - 2 = 0$

Note that quadratic functions are a special type of polynomial.

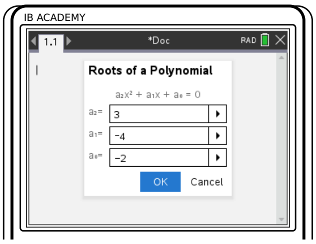


Press menu, choose
`3: Algebra`
`3: Polynomial Tools`
`1: Find Roots of Polynomial`



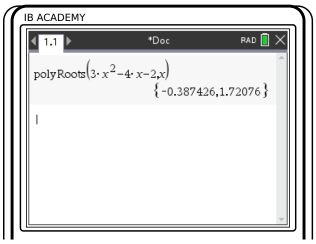
Find Roots of a Polynomial
Degree: 2
Roots: Real

`Degree = 2`, `Roots = Real`, press OK



Roots of a Polynomial
$a_2x^2 + a_1x + a_0 = 0$
a_2= 3
a_1= -4
a_0= -2

Enter values `a2`, `a1` and `a0`. Press OK



polyRoots$(3\cdot x^2-4\cdot x-2,x)$
$\{-0.387426, 1.72076\}$

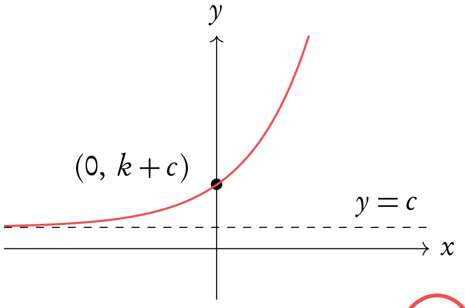
so $x = 1.72$ or $x = -0.387$

2.5 Exponential models

Further, you need to be familiar with exponential functions. An exponential function is one where the variable (e.g. x) is in the power itself.
In questions dealing with exponential functions, you will need to know how to describe their asymptotes and y-intercepts. You can use the components of a function's equation to find these.

$$y = ka^{rx} + c$$

with
horizontal asymptote at $y = c$
$m = y$-intercept at $(0,\ k+c)$

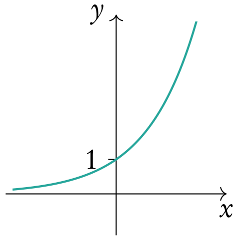


Asymptote a line that a graph approaches but never quite touches.

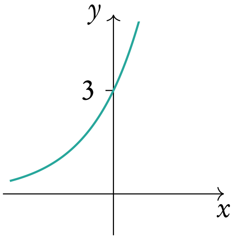
Use GDC to sketch more complicated functions.

Example.

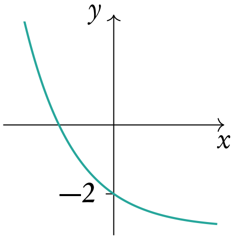
$y = 2^x$



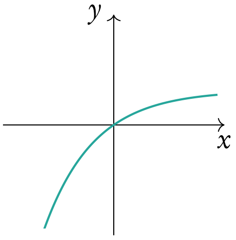
$y = 3 \cdot (2^x)$



$y = 2^{-x} - 3$



$y = -2^{-x} + 1$



Solving exponential functions

A population of rabbits is modelled by the function $P(t) = 24e^{0.19t}$, $t \geqslant 0$, where P is the population of rabbits and t is the time in months. After how many months does the population reach 1 million?

1.	Set up an equation you are looking to solve.	$1000000 = 24e^{0.19t}$
2.	Plot both sides of the equation as separate functions on your GDC.	$y_1 = 1000000$ $y_2 = 24e^{0.19t}$
3.	Find the x-coordinate of the intersection point.	$x = 55.987$ $\Rightarrow$ 56 months

Remember that e is just a number (2.71828...), so treat it like any other number.

You get points on your IB exam for writing out an equation like this!

2.6 Composite and inverse functions

2.6.1 Composite functions

Composite functions are a combination of two functions.

$$(f \circ g)(x) \qquad \text{means } f \text{ of } g \text{ of } x$$

To find the composite function above substitute the function of $g(x)$ into the x of $f(x)$.

Remember
$f \circ g(x) \neq g \circ f(x)$

Example.

Let $f(x) = 2x + 3$ and $g(x) = x^2$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$(f \circ g)(x)$: replace x in the $f(x)$ function with the entire $g(x)$ function

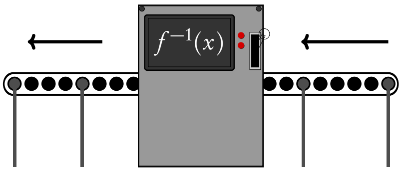
$$(2g(x)) + 3 = 2x^2 + 3$$

$(g \circ f)(x)$: replace x in the $g(x)$ function with the entire $f(x)$ function

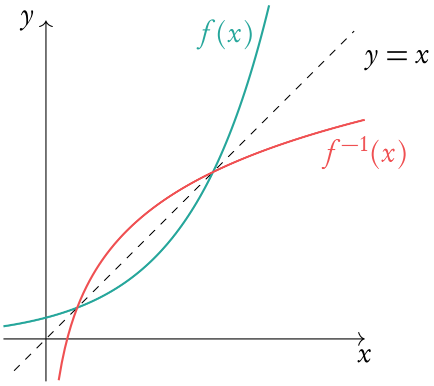
$$\big(f(x)\big)^2 = (2x+3)^2$$

2.6.2 Inverse functions, $f^{-1}(x)$

Inverse functions are the reverse of a function. Finding the input x for the output y. You can think of it as going backwards through the number machine.



This is the same as reflecting a graph in the $y = x$ axis.



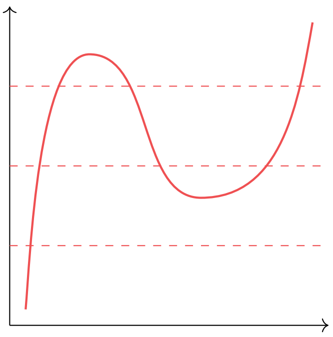
Finding the inverse function

$f(x) = 2x^3 + 3$, find $f^{-1}(x)$

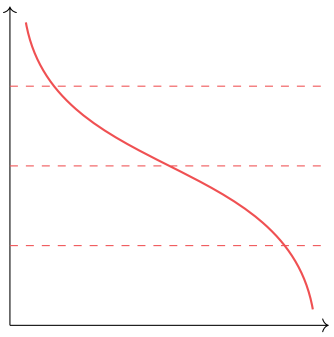
1.	Replace $f(x)$ with y.	$y = 2x^3 + 3$
2.	Solve for x.	$y - 3 = 2x^3$ $\Rightarrow \quad \frac{y-3}{2} = x^3$ $\Rightarrow \quad \sqrt[3]{\frac{y-3}{2}} = x$
3.	Replace x with $f^{-1}(x)$ and y with x.	$\sqrt[3]{\frac{x-3}{2}} = f^{-1}(x)$

For an inverse function, $f^{-1}(x)$, to exist the function $f(x)$ must be a one-to-one function; each input has a unique output. A function is one-to-one if horizontal lines drawn across it pass through the function only once. Sometimes to make an inverse function, the domain of a function needs to be restricted so that it is a one-to-one function.

many-to-one



one-to-one



2.7 Graph transformations

By adding and/or multiplying by constants we can transform a function into another function.

Change to $f(x)$	Effect
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $1/a$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis
$f(x) + a$	Move graph a units upwards
$f(x + a)$	Move graph a units to the left

Exam hint: describe the transformation with words as well to guarantee marks.

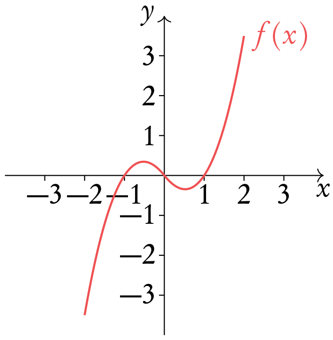
Always do translations last

Some transformations should be completed before others. A horizontal shift comes before a horizontal stretch, but a vertical stretch comes before a vertical shift.

Transforming functions $f(x) \rightarrow af(x+b)$

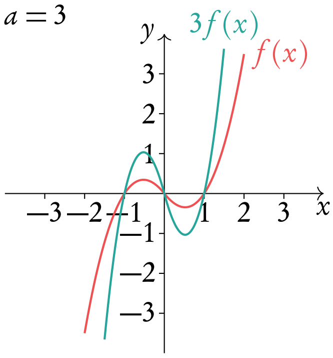
Given $f(x)=\frac{1}{4}x^3+x^2-\frac{5}{4}x$, draw $3f(x-1)$.

1. Sketch $f(x)$.



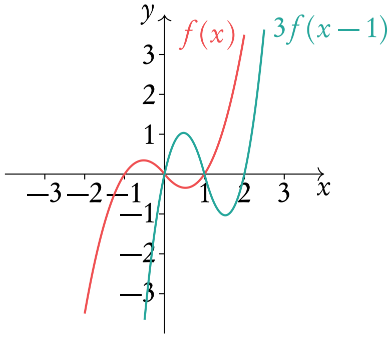
2. Stretch the graph by the factor of a.

$a = 3$



3. Move graph by $-b$.

Move graph by 1 to the *right*



2.8 Modelling

2.8.1 Sinusoidal models

Trigonometric functions, *sine* and *cosine*, make sinusoidal shapes when graphed.

As with other functions, the height, width and position on the axes of a trigonometric function is determined by its parameters. For sinusoidal models we describe these parameters with special names; the vertical stretch is determined by the *amplitude*, the horizontal stretch by the *period*, the horizontal shift by the *phase shift*, and an upward/downward shift by the *position of the principle axis*. When working with sinusoidal models assume that they are in radians, unless the question states otherwise.

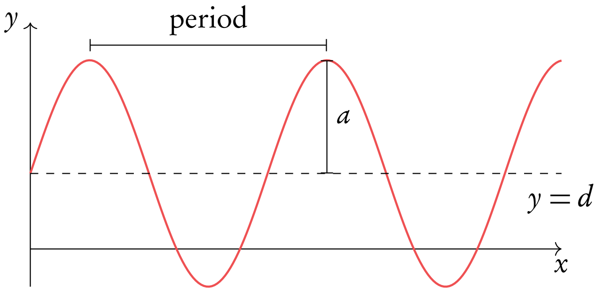
$$y = a\sin\big(b\,(x-c)\big)+d$$

amplitude of a

principle axis at $y=d$

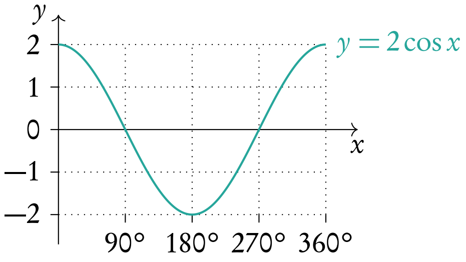
period of $\dfrac{360°}{b}=\dfrac{2\pi}{b}$

phase shift of c



Example.

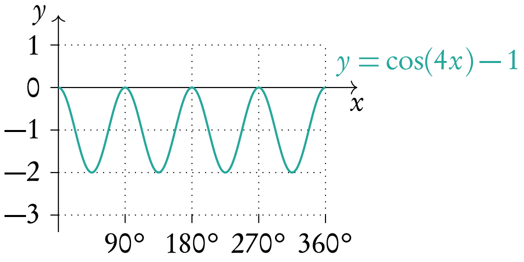
Transformations of $y=\cos x$.



amplitude $=2$

principle axis at $y=0$

period $=\dfrac{360°}{1}=360°$



amplitude $=1$

principle axis at $y=-1$

period $=\dfrac{360°}{4}=90°$

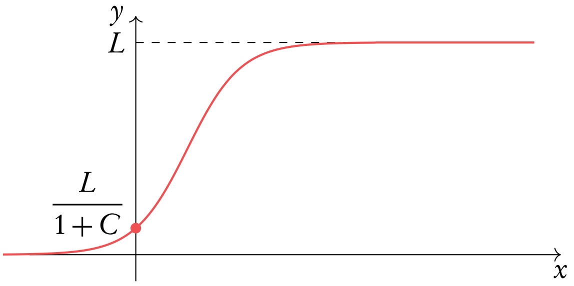
2.8.2 Logistic models

Logistic models are used in situations where there is a restriction on growth. For example, the maximum height a person can grow to or the maximum population size an animal population can reach. A horizontal asymptote occurs at $f(x) = L$. This value, L, is known as the *carrying capacity*. The parameter k measures how fast the model is growing.

$$f(x) = \frac{L}{1 + Ce^{-kx}}, \quad L, C, k > 0$$

horizontal asymptote at $y = L$

y-intercept at $y = \dfrac{L}{1+C}$



Solving logistic functions

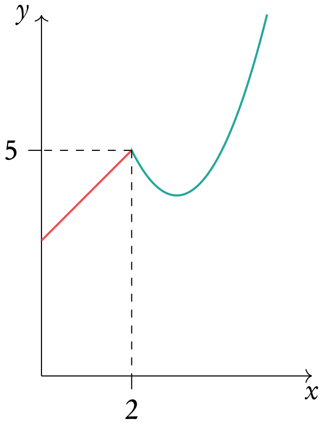
Twenty birds are introduced to an empty island. The island's carrying capacity is 10 000. After 3 years there are 35 birds. Determine the logistic model and use it to estimate the bird population after 5 years.

1.	Find L.	L is the carrying capacity of the island: $L = 10000$
2.	Use L to find C.	The initial population is equal to $\dfrac{L}{1+C}$ $\dfrac{L}{1+C} = 20$ $\dfrac{10000}{1+C} = 20$ $1 + C = \dfrac{10000}{20}$ $C = 500 - 1 = 499$
3.	Fill the values of L and C into the logistic function.	$f(x) = \dfrac{10000}{1 + 499e^{-kx}}$
4.	Use known values given in the question to find k. Solve using your calculator.	In our model we know that $f(3) = 35$. We can plot this function to find k. $f(3) = \dfrac{10000}{1 + 499e^{-3k}} = 35$ $k = 0.187$
5.	Fill in the values for L, C, and k into the logistic growth function.	$f(x) = \dfrac{10000}{1 + 499e^{-0.187x}}$
6.	Plot the function and find its value at the specific x-value.	The question asks to estimate the bird population after 5 years, so $x = 5$: $f(5) = 50.79 \approx 51$

2.8.3 Piecewise models

Sometimes data does not follow one of the previously mentioned models perfectly. In these cases a piecewise model may be used in order to fit the data to a model. Piecewise models are made up of multiple functions which are restricted to certain domains to create a connected function.

$$f(x)=\begin{cases} x+3 & 0<x\leq 2 \\ (x-3)^2+4 & 2<x<6 \end{cases}$$



Making piecewise functions continuous

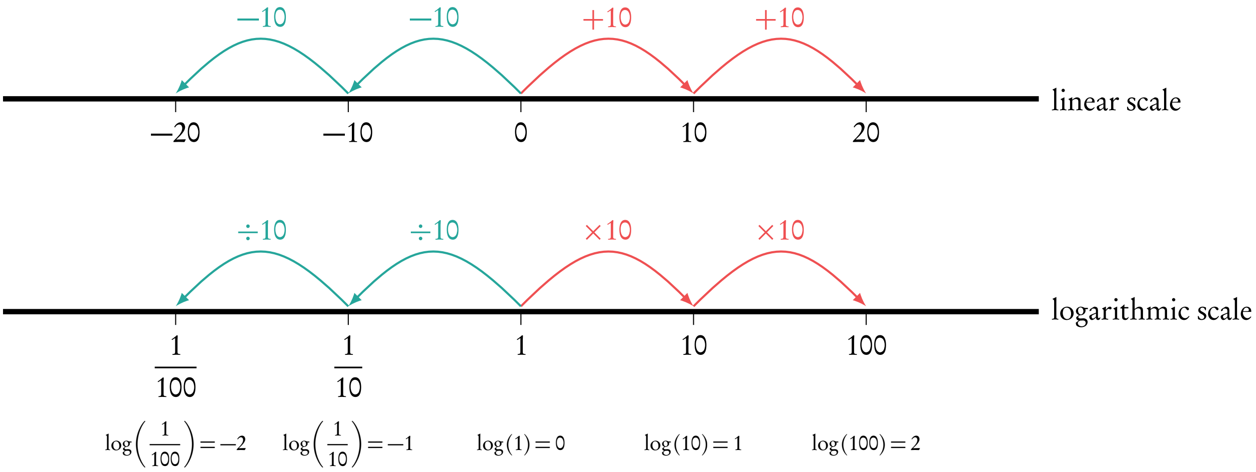
Find the value of a such that the following piecewise function is continuous.

$$f(x)=\begin{cases} 1+x & 0\leq x<2 \\ ax^2+x & x\geq 2 \end{cases}$$

1.	If a piecewise function is continuous, the value of each sub function is the same at the point they meet. First, find this point.	The point occurs at $x=2$.
2.	Find the value of both functions at this point.	$f(2)=1+2=3$ $f(2)=a(2)^2+2=4a+2$
3.	Set these equal to one another and solve for the missing value.	$3=4a+2$ $a=\frac{1}{4}$

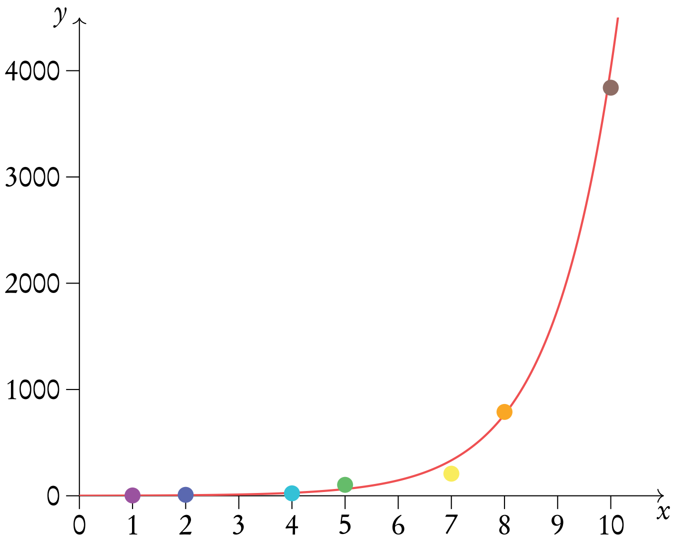
2.9 Scaling graphs

Logarithms allow us to display data which has a large range of values in a more compact way. On a linear number line, moving a step to the right means adding a number, but on a logarithmic scale moving a step to the right means multiplying by a number. Taking the logarithm of a log scale gives a linear scale.

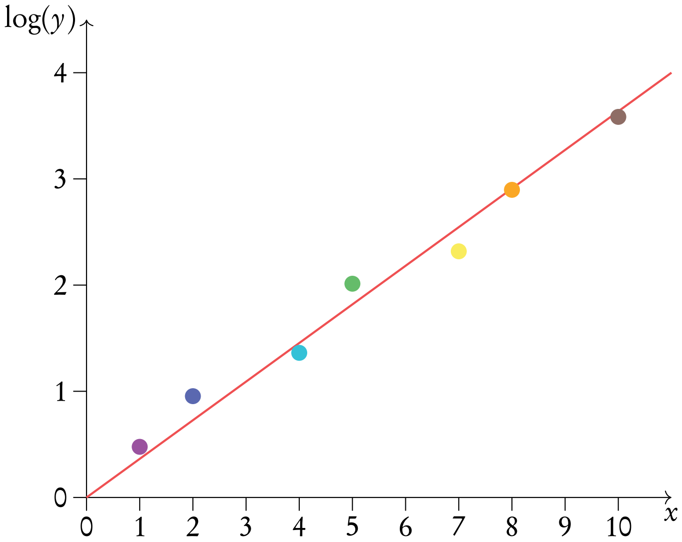


Taking the log of data set y makes the values a similar scale to those of the data set x. The data sets x and y have an exponential relationship, so the data sets x and $\log(y)$ have a linear relationship. To find the values of the y data points from the $\log(y)$ graph, first read off their value on the $\log(y)$ graph. Then raise 10, or the base of the log, to the power of this value. This reverses the $\log(y)$ since it is a log of base 10.

x	y
1	3
2	9
4	23
5	103
7	208
8	789
10	3840



x	$\log(y)$
1	0.477
2	0.954
4	1.362
5	2.013
7	2.318
8	2.897
10	3.584



When we take the log of one axes the graph is called a semi-log graph, when we take the log of both axes the graph is called a log-log graph. The IB expects you to interpret log-log and semi-log graphs, but not draw them yourself.

TRIGONOMETRY

3

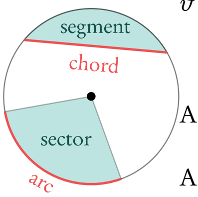
Table of contents & cheatsheet

3.1. Basic trigonometry 54

radians $= \frac{\pi}{180°} \times$ degrees degrees $= \frac{180°}{\pi} \times$ radians

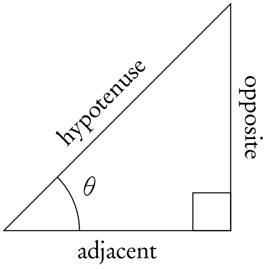
Before each question make sure calculator is in correct setting: degrees or radians?

$\vartheta =$ angle, $r =$ radius



	ϑ in degree	ϑ in radians
Arc length =	$\frac{\vartheta}{360} \times 2\pi r$	$\vartheta \times r$
Area of a sector =	$\frac{\vartheta}{360} \times \pi r^2$	$\frac{1}{2} \times \vartheta \times r^2$

Right-angle triangle (triangle with 90° angle)

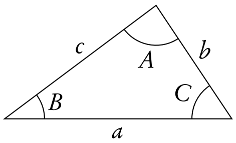


$\sin\vartheta = \frac{\text{opposite}}{\text{hypotenuse}}$ **SOH**

$\cos\vartheta = \frac{\text{adjacent}}{\text{hypotenuse}}$ **CAH**

$\tan\vartheta = \frac{\text{opposite}}{\text{adjacent}}$ **TOA**

Non-right angle triangles



DB 3.4

Sine rule: $\frac{a}{\sin A} = \overset{b}{\frac{b}{\sin B}} = \frac{c}{\sin C}$

Use this rule when you know: 2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides).

Cosine rule: $c^2 = a^2 + b^2 - 2\,ab\cos C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

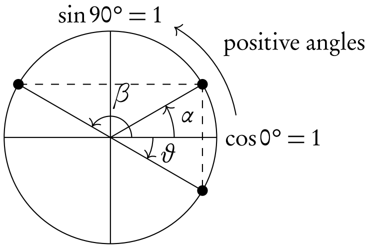
Area of a triangle: Area $= \frac{1}{2}ab\sin C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

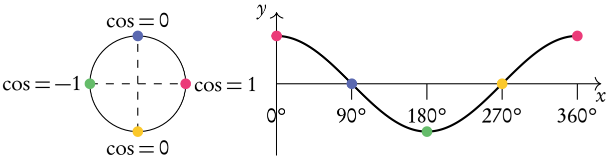
Three-figure bearings

Direction given as an angle of a full circle. North is 0° and the angle is expressed in the clockwise direction from North. So East is 90°, South is 180° and West 270°.

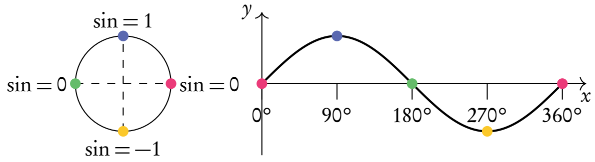
3.2. Circular functions 58



Cos graph from unit circle



Sine graph from unit circle



Trigonometric function $y = a\sin(bx + c) + d$

Amplitude: a

Period: $\frac{360°}{b}$ or $\frac{2\pi}{b}$

Horizontal shift: c

Vertical shift: d

Trigonometric identities

$\tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta}$

$\sin^2\vartheta + \cos^2\vartheta = 1$

3.1 Basic fundamentals

3.1.1 Radians

$$\text{radians} = \frac{\pi}{180^\circ} \times \text{degrees}$$

$$\text{degrees} = \frac{180^\circ}{\pi} \times \text{radians}$$

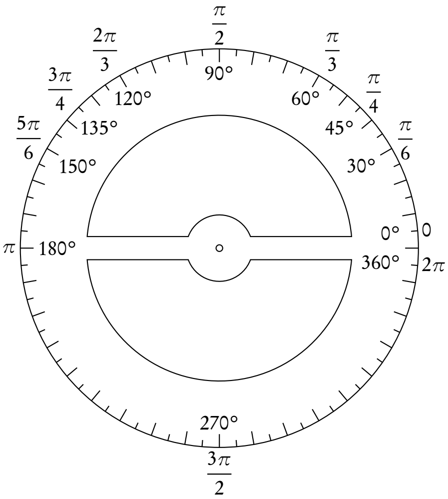
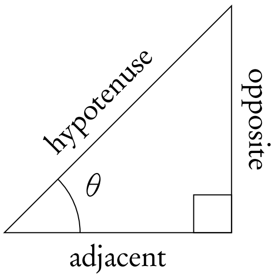


Table 3.1: Common radians/degrees conversions

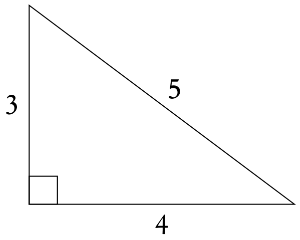
Degrees	0°	30°	45°	60°	90°	120°	135°	180°	270°	360°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

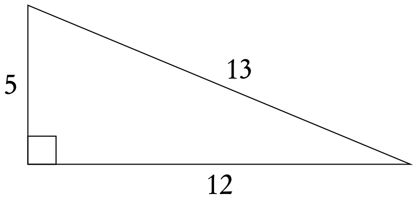
3.1.2 Right-angle triangles

$a^2 = b^2 + c^2$	Pythagoras
$\sin\vartheta = \frac{\text{opposite}}{\text{hypotenuse}}$	SOH
$\cos\vartheta = \frac{\text{adjacent}}{\text{hypotenuse}}$	CAH
$\tan\vartheta = \frac{\text{opposite}}{\text{adjacent}}$	TOA

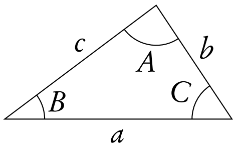


The following two right angle triangles with whole numbers for all the sides come up often in past exam questions.





3.1.3 Non-right angle triangles



To find any missing angles or side lengths in non-right angle triangles, use the *cosine* and *sine* rule. Remember that the angles in the triangle add up to 180°!

Read the question: does it specify if you are looking for an acute (less than 90°) or obtuse (more than 90°) angle? If not there may be 2 solutions. Exam hint: Use sketches when working with worded questions!

DB 3.2

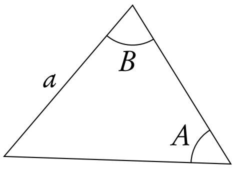
Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

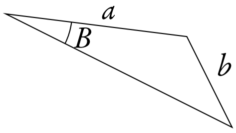
Use this rule when you know:

2 angles and a side
(not between the angles)

or

2 sides and an angle
(not between the sides)





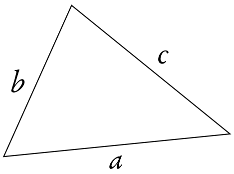
Cosine rule: $c^2 = a^2 + b^2 - 2\,ab\cos C$

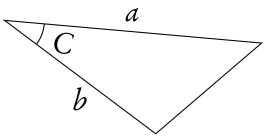
Use this rule when you know:

3 sides

or

2 sides and the angle between them

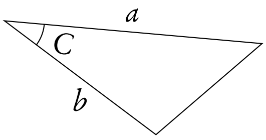




Area of a triangle: Area $= \frac{1}{2}ab\sin C$

Use this rule when you know:

2 sides and the angle between them



Example.

$\triangle ABC$: $A = 40°$, $B = 73°$, $a = 27$ cm.

Find $\angle C$

$$\angle C = 180° - 40° - 73° = 67°$$

Find b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{27}{\sin 40°} = \frac{b}{\sin 73°}$$
$$b = \frac{27}{\sin 40°} \cdot \sin 73° = 40.169 \approx 40.2\,\text{cm}$$

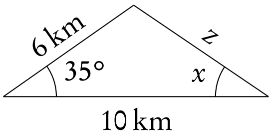
Find c

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$c = \frac{27}{\sin 40°} \times \sin 67° = 38.7\,\text{cm}$$

Find the area

$$\text{Area} = \frac{1}{2} \cdot 27 \cdot 40 \cdot 2 \cdot \sin 67°$$
$$= 499.59 \approx 500\,\text{cm}^2$$

Example.



Find z

$$z^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 35°$$
$$z^2 = 37.70$$
$$z = 6.14\,\text{km}$$

Find $\angle x$

$$\frac{6}{\sin x} = \frac{6.14}{\sin 35°}$$
$$\sin x = 0.56$$
$$x = \sin^{-1}(0.56) = 55.91°$$

3.1.4 Ambiguous case

Ambiguous case, also known as an angle-side-side case, is when the triangle is not unique from the given information. It happens when you are given two sides and an angle not between those sides in a triangle.

You have to use a sine rule to solve a problem in this case. However, one needs to remember that $\sin x = \sin(180° - x)$, meaning that your answer for an angle is not just x, but also $180° - x$.

In other words, we might get two different possible angles as an answer and thus two different possible triangles that satisfy the information given.

However, that is not always the case, if the sum of the two known angles becomes bigger than 180°. So if you are required to calculate the third angle or total area of a triangle, you might have to do the calculations for two different triangles using both of your angles.

Example.

$\triangle ABC$: $B = 33°$, $a = 23\,\text{cm}$, $b = 14\,\text{cm}$.

Find $\angle A$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

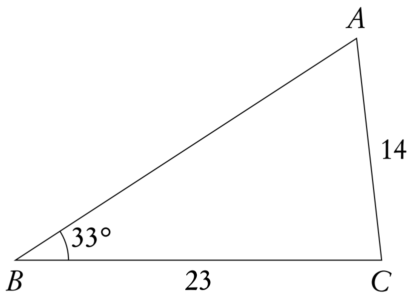
$$\frac{14}{\sin 33°} = \frac{23}{\sin A}$$

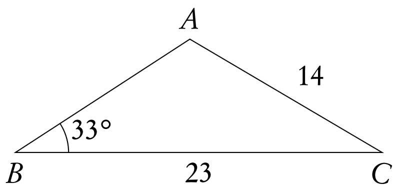
$$\angle A_1 = 63.5°$$

$$\angle A_2 = 180° - 63.5° = 117°$$

$$\angle A_2 + 33° < 180° \text{ thus also a possible angle}$$

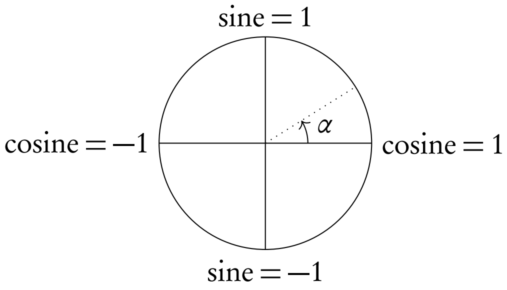
Draw the two possible triangles.





3.2 Circular functions

3.2.1 Unit circle



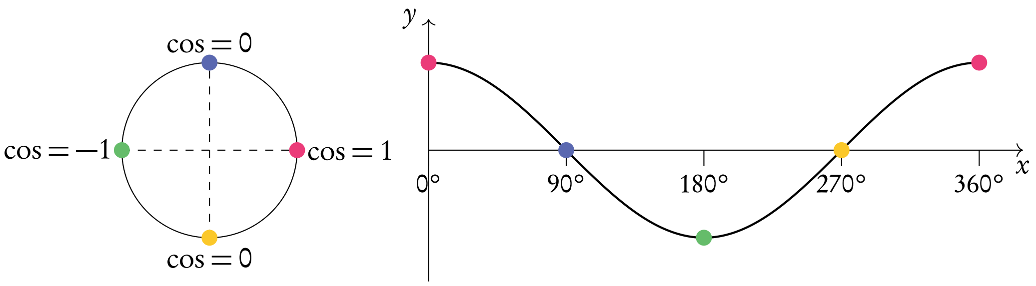
The unit circle is a circle with a radius of 1 drawn from the origin of a set of axes. The y-axis corresponds to *sine* and the x-axis to *cosine*; so at the coordinate $(0, 1)$ it can be said that cosine $= 0$ and sine $= 1$, just like in the $\sin x$ and $\cos x$ graphs when plotted.

The unit circle is a a tool that you can use when solving problems involving circular functions. You can use it to find all the solutions to a trigonometric equation within a certain domain.

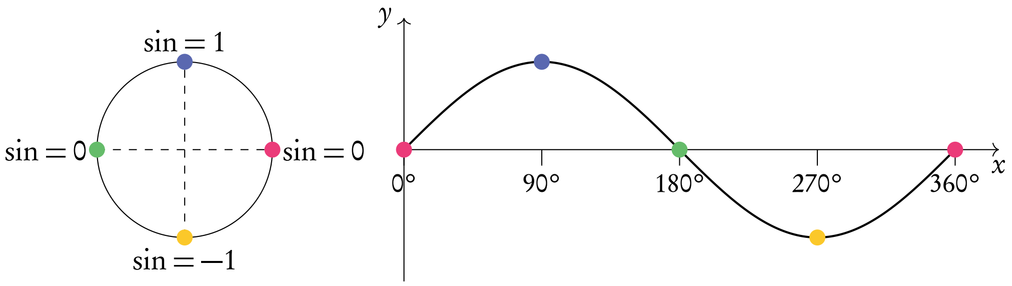
As you can see from their graphs, functions with $\sin x$, $\cos x$ or $\tan x$ repeat themselves every given period; this is why they are also called *circular functions*. As a result, for each y-value there is an infinite amount of x-values that could give you the same output. This is why questions will give you a set domain that limits the x-values you should consider in your calculations or represent on your sketch (e.g. $0° \leq x \leq 360°$).

The unit circle can be used to construct the graphs of $\sin x$ and $\cos x$. This is done by selecting a few angles on the unit circle and reading off the corresponding values of cos or sin of these angles. Then these points are plotted on a graph and can be connected.

Cos graph from unit circle



Sine graph from unit circle



3.3 Trigonometric relations

In order to solve trigonometric equations, you will sometimes need to use identities. Identities allow you to rewrite your equation in a way that will make it easier to solve algebraically.

DB 3.8

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$

Using GDC to solve trigonometric equations

Solve the equation $\cos^2(x) + 4\sin(4x) = 1$ for $\pi < x < 2\pi$.

1.	Rearrange the equation so that it equals zero.	$\cos^2(x) + 4\sin(4x) - 1 = 0$
2.	Plot the function on your calculator. Check whether your calculator should be in degrees or radians.	
3.	Change your v-window to show only the x values which the question asks for.	In this case we only look at x values between π and 2π.
4.	Find the roots of the function.	$x = 3.142, 3.898, 4.775, 5.464, 6.283$
5.	Check that all solutions are within the interval given.	$3.142 = \pi, 6.283 = 2\pi$, so we do not include these solutions, since x must be larger than π and smaller than 2π.
6.	Write out all solutions within the interval.	$x = 3.898, 4.775, 5.464$

3.3.1 Geometric transformations

Matrices can be used to transform a point, when the point is written as a vector. The matrix multiplies the vector to give a new point. These matrices are known as transformation matrices and are used to perform various transformations. These transformations include reflections, horizontal and vertical stretches, enlargements, translations, and rotations. Transformation matrices can also be combined to perform multiple transformations.

Example.

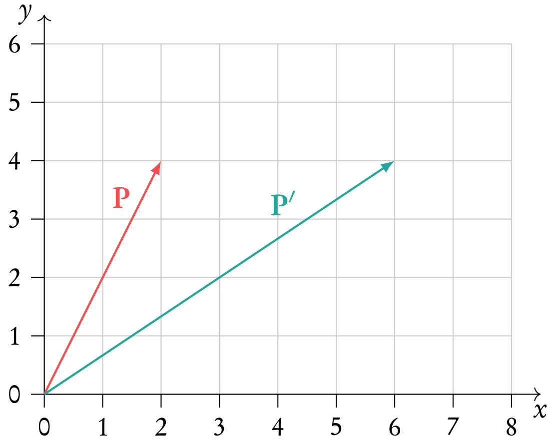
Matrix transformation

$$\mathrm{P} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

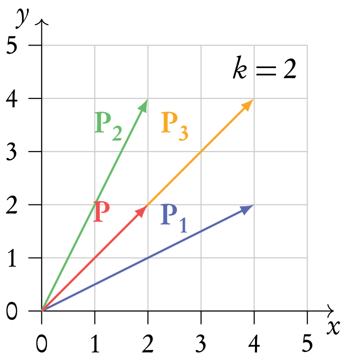
$$\mathbf{M}\mathrm{P} = \mathrm{P}'$$

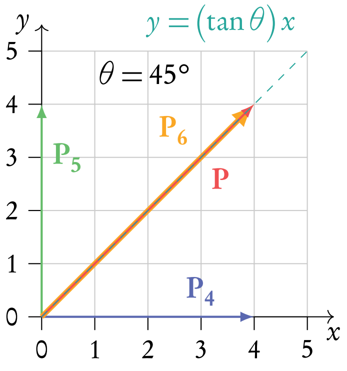
$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



There are many standard matrices for performing different types of transformations. These are given to you in your data booklet.

$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	stretch parallel to the x-axis with a scale factor of k	(P_1)
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	stretch parallel to the y-axis with a scale factor of k	(P_2)
$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	enlargement with a scale factor of k centered at $(0,0)$	(P_3)
$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$	clockwise rotation of angle θ about the origin, $\theta > 0$	(P_4)
$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	counter-clockwise rotation of angle θ about the origin, $\theta > 0$	(P_5)
$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$	reflection in the line $y = (\tan\theta)\,x$	(P_6)





3.3.2 Determinant meaning

Transformation matrices can be applied to shapes and vectors as well as points. When transforming a shape with a transformation matrix, the determinant of the matrix can tell us about the area of the transformed shape. The transformed shape is sometimes called the image.

$$\text{area of image} = |\det(\mathbf{A})| \times \text{area of object}$$

Example.

A triangle has area 45 cm^2. It is transformed using the matrix **A**.

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

The determinant of matrix **A** is $3 \times 3 - 0 \times 0 = 9$. The area of the transformed triangle, or image is therefore $3 \times 45 = 135\,\text{cm}^2$.

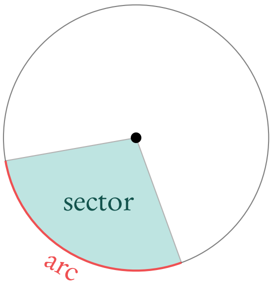
3.4 Circles

3.4.1 Circle formulas

DB 3.4

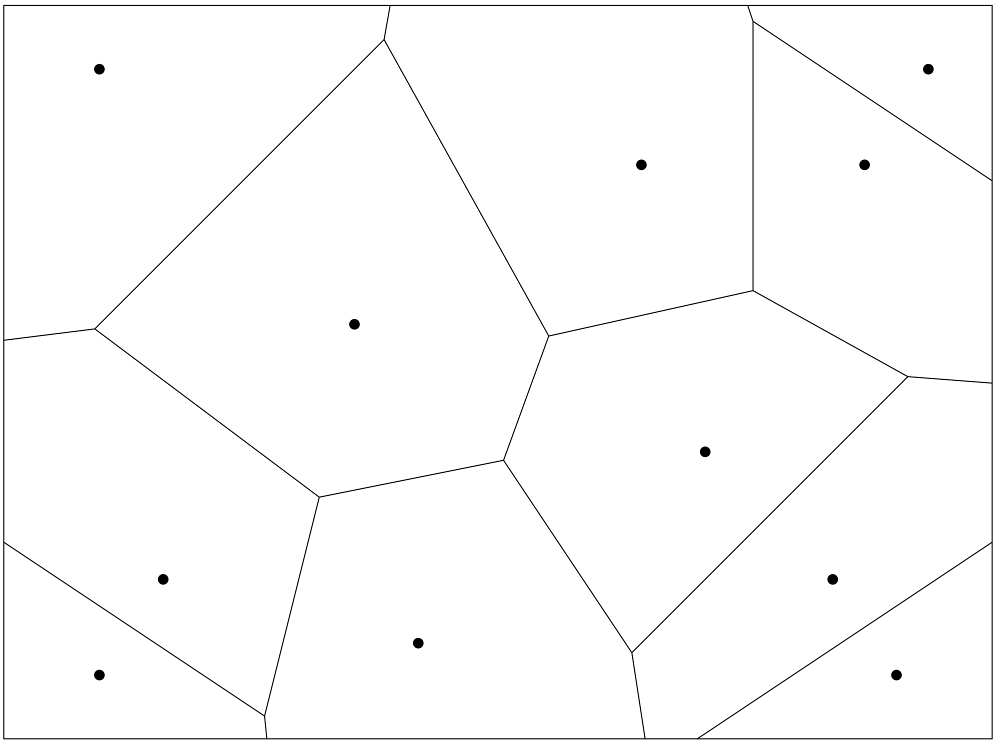
$\vartheta = \text{angle}$, $r = \text{radius}$

	ϑ in degree	ϑ in radians
Arc length =	$\frac{\vartheta}{360} \times 2\pi r$	$\vartheta \times r$
Area of a sector =	$\frac{\vartheta}{360} \times \pi r^2$	$\frac{1}{2} \times \vartheta \times r^2$



3.5 Voronoi diagrams

With a Voronoi diagram you can divide a plane into regions based on a set of sites in it. The partitioning is made based minimal distance to the sites.



Site	a point on a plane
Cell	a region containing all the points for which the enclosed site is the closest one. Each cell encloses a single site
Edge	a boundary between two cells
Vertex	a meeting point of two or more edges

3.5.1 Nearest neighbour interpolation

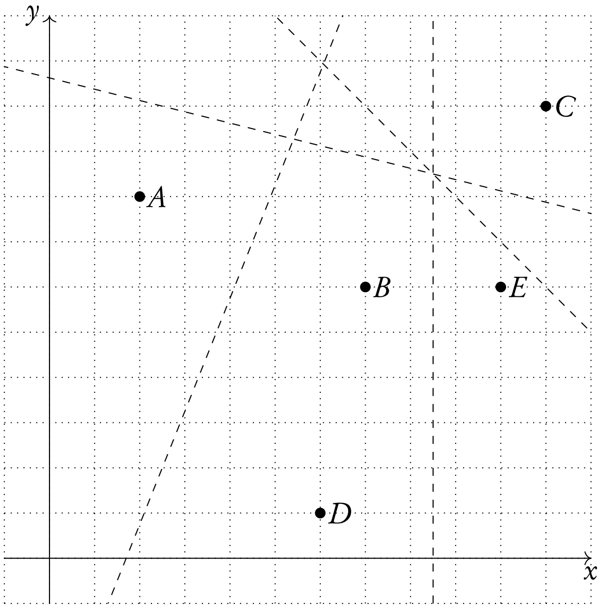
To construct a Voronoi diagram based on a set of sites, you need to find the boundaries between the cells that will enclose each of them. These edges lie along the perpendicular bisectors between neighbouring sites. Finding these perpendicular bisectors is referred to as a form of nearest neighbour interpolation.

Perpendicular bisector the line passing through the midpoint between two points and at a 90°angle to the line segment that connects them

Nearest neighbour interpolation finding the site closest to any given point

Voronoi diagrams

Points $A(2,8)$, $B(7,6)$, $C(11,10)$, $D(6,1)$ and $E(10,6)$ represent restaurants of a pizza chain in a city.



(a) Calculate the gradient of the line connecting points B and D.

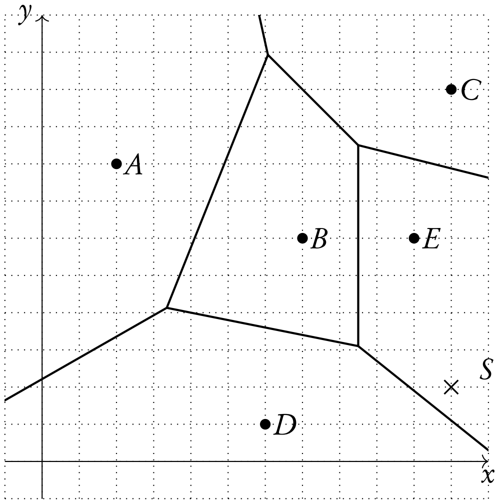
(b) Hence, find the equation of the perpendicular bisector of points B and D.

(c) Sketch the completed Voronoi diagram.

To optimise delivery, pizza orders are always distributed to the closest restaurant.

(d) Sam lives at point $S(11,2)$. Which pizza place will prepare Sam's pizza?

A perpendicular bisector of two points always passes through their midpoint

1.	Use gradient formula to find m.	$m=\frac{6-1}{7-6}=5$
2.	Find gradient of perpendicular line.	$m_2=\frac{-1}{5}$ $\Rightarrow y=\frac{-1}{5}x+c$
3.	Find one point that lies on the perpendicular bisector.	Midpoint of B and D: $\left(\frac{7+6}{2},\frac{6+1}{2}\right)=(6.5,3.5)$
4.	Use point to find c.	$3.5=\frac{-1}{5}(6.5)+c$ $c=4.8$ $\Rightarrow y=\frac{-1}{5}x+4.8$
5.	Sketch.	
6.	Use Voronoi diagram to determine which cell a given point falls into.	The pizza place closest to Sam is E, therefore Sam's order would be prepared at and delivered from E.

4

VECTORS AND GRAPH THEORY

Table of contents & cheatsheet

Definitions

Vector	a geometric object with *magnitude* (length) and *direction*, represented by an *arrow*.
Collinear points	points that lie on the same line
Unit vector	vector with magnitude 1
Base vector	$\vec{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$

Working with vectors

Vector from point O to point A: $\vec{OA} = \vec{a} = \begin{pmatrix}3\\2\end{pmatrix}$

Vector from point O to point B: $\vec{OB} = \vec{b} = \begin{pmatrix}-1\\1\end{pmatrix}$

Can be written in two ways:

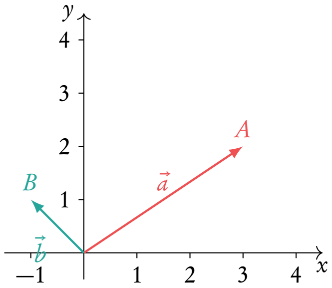
$$\vec{a} = \begin{pmatrix}3\\2\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

$$\vec{a} = 3i + 2j + 0k = 3i + 2j$$

Length of $\vec{a}$: $|\vec{a}| = \sqrt{x^2+y^2} = \sqrt{3^2+2^2} = \sqrt{13}$

Addition & multiplication: $\vec{a} + 2\vec{b} = \begin{pmatrix}3\\2\end{pmatrix} + 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} + \begin{pmatrix}-2\\2\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$

Subtraction: $\vec{a} - \vec{b} = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$



Equations of lines

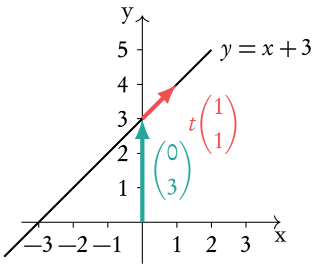
Example of a line:

$$r = \begin{pmatrix}0\\3\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}$$

direction vector

parameter

position vector



Dot product

The dot product of two vectors $\vec{c} \cdot \vec{d}$ can be used to find the angle between them.

Let $\vec{c} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}, \vec{d} = \begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$:

$$\vec{c} \cdot \vec{d} = |\vec{c}||\vec{d}|\cos\vartheta$$

$$\vec{c} \cdot \vec{d} = c_1 d_1 + c_2 d_2 + c_3 d_3$$

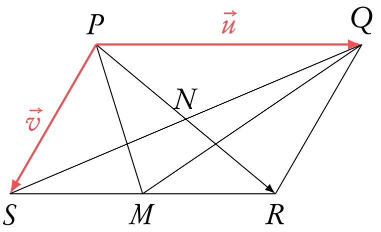
4.1 Vector fundamentals

Vectors are a geometric object with a *magnitude* (length) and *direction*. They are represented by an *arrow*, where the arrow shows the direction and the length represents the magnitude.

So looking at the diagram we can see that vector $\vec{u}$ has a greater magnitude than $\vec{v}$. Vectors can also be described in terms of the points they pass between. So

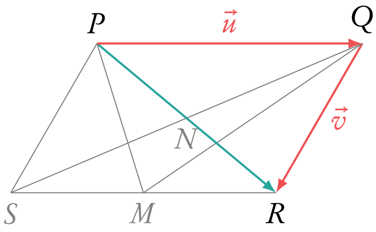
$$\begin{cases}\vec{u}=\vec{PQ}\\ \vec{v}=\vec{PS}\end{cases}$$

with the arrow over the top showing the direction.

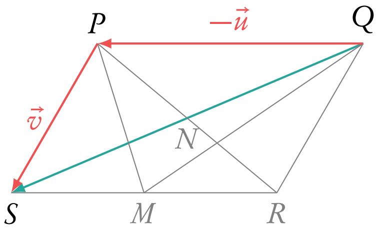


You can use vectors as a geometric algebra, expressing other vectors in terms of $\vec{u}$ and $\vec{v}$. For example

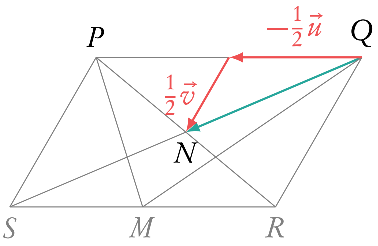
$\vec{PR}=\vec{u}+\vec{v}$



$\vec{QS}=-\vec{u}+\vec{v}$



$\vec{QN}=\frac{1}{2}(-\vec{u}+\vec{v})$



This may seem slightly counter-intuitive at first. But if we add in some possible figures you can see how it works. If $\vec{u}$ moves 5 units to the left and $\vec{v}$ moves 1 unit to the right (—left) and 3 units down.

Then $\vec{PR}=\vec{u}+\vec{v}=$ 5 units to the left —1 unit to the right and 3 units down $=$ 4 units to the left and 3 units down.

4.1.1 Vectors with value

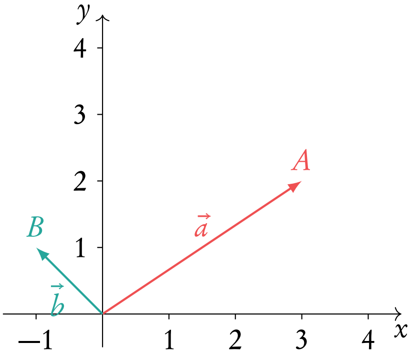
Formally the value of a vector is defined by its direction and magnitude within a 2D or 3D space. You can think of this as the steps it has to take to go from its starting point to its end, moving only in the x, y and z axis.

Vector from point O to point A:

$$\vec{OA} = \vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Vector from point O to point B:

$$\vec{OB} = \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



Vectors can be written in two ways:

Note: unless told otherwise, answer questions in the form used in the question.

1. $\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where the top value is movement in the x-axis. Then the next is movement in the y and finally in the z. Here the vector is in 2D space as there is no value for the z-axis.
2. as the sum of the three base vectors:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Here $\vec{i}$ is moving 1 unit in the x-axis, $\vec{j}$ 1 unit in the y-axis and $\vec{k}$ 1 unit in the z-axis.

$$\vec{a} = 3i + 2j + 0k = 3i + 2j$$

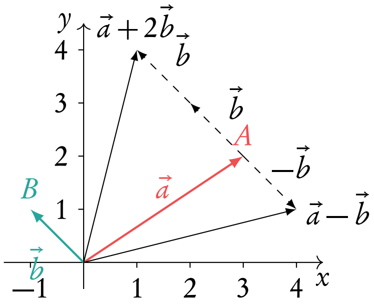
When we work with vectors we carry out the mathematical operation in each axis separately. So x-values with x-values and so on.

Addition & multiplication:

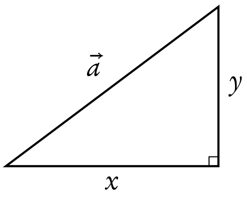
$$\vec{a} + 2\vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Subtraction:

$$\vec{a} - \vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



However it must be remembered that vector notation does not give us the actual length (magnitude) of the vector. To find this we use something familiar.



Length of $\vec{a}$:

$$|\vec{a}| = \sqrt{x^2 + y^2}$$
$$= \sqrt{3^2 + 2^2} = \sqrt{13}$$

Sometimes you will be asked to work with unit vectors. These are vectors with a magnitude of 1. We can convert all vectors to unit vectors.

Determine the unit vector $\hat{a}$ in the direction of any vector $\vec{a}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}$$

You need to know how to re-scale vectors. Re-scaling refers to changing a vector's magnitude without changing its direction. Turning a vector into a unit vector is an example of re-scaling a vector. The concept of velocity might come up in these questions. The velocity of an object refers to its speed and direction. It is described by a vector with magnitude equal to the object's speed and which points in the direction of the velocity of the object.

Re-scaling vectors

Find the velocity of a particle with speed 7 m s^{-1} in the direction $3\mathbf{i} + 4\mathbf{j}$.

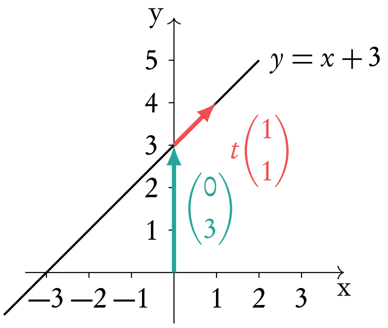
1.	Calculate the magnitude of the direction vector.	magnitude $= \sqrt{3^2 + 4^2} = \sqrt{25} = 5$
2.	Divide the vector by its magnitude to find a unit vector.	$\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
3.	Multiply the unit vector by the required scalar value.	Here, we multiply by the speed to find the velocity: $\mathbf{v} = \frac{3 \times 7}{5}\mathbf{i} + \frac{4 \times 7}{5}\mathbf{j}$ $\mathbf{v} = \frac{21}{5}\mathbf{i} + \frac{28}{5}\mathbf{j}$

We can further divide vectors into two types:

position vectors	vectors from the origin to a point, e.g. $P = (-1, 3) \Rightarrow \vec{P} = \begin{pmatrix}-1\\3\end{pmatrix}$.
direction vectors	vectors that define a direction.

Using both we can define lines in terms of vectors.
Example of a line:

$$r = \underbrace{\begin{pmatrix}0\\3\end{pmatrix}}_{\text{position vector}} + \underbrace{t}_{\text{parameter}} \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\text{direction vector}}$$



Note the position vector can go to any where on the line. So in this example we could also use $(-3, 0)$ or $(1, 4)$. Equally the direction vector can be scaled. So we could use $(2, 2)$, $(30, 30)$, ...

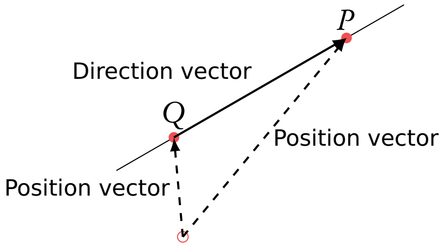
Because of this parallel lines will have direction vectors with the same ratio but not necessarily in exact numbers.

Parallel lines: direction vector of L_1 = direction vector of $L_2 \times$ constant

Questions often deal with points and or multiple lines. It is worth making a sketch to help understand the question.

4.1.2 Converting to the parametric form

Checking whether a point lies on a line



Find the equation of the line passing through points $P = (1,3,2)$ and $Q = (0,-1,4)$. Does point $R = (-2,9,1)$ lie on the line?

Note this can go either way from Q to P of P to Q.

1. Write points as position vectors.

$$\vec{P} = \begin{pmatrix}1\\3\\2\end{pmatrix}, \vec{Q} = \begin{pmatrix}0\\-1\\4\end{pmatrix}$$

2. Set the direction vector equal to the vector between the points.

$$\begin{pmatrix}0-1\\-1-3\\4-2\end{pmatrix} = \begin{pmatrix}-1\\-4\\2\end{pmatrix}$$

3. Choose either of the points as a position vector.

Take P for example: $r = \begin{pmatrix}1\\3\\2\end{pmatrix} + t\begin{pmatrix}-1\\-4\\2\end{pmatrix}$

4. Equate desired point with the line equation. If there is no contradiction, the point lies on the line.

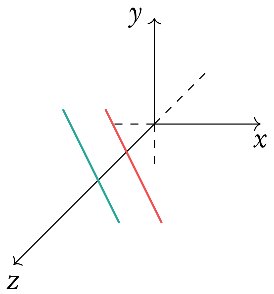
$$R = r: \begin{pmatrix}-2\\9\\1\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix} + t\begin{pmatrix}-1\\-4\\2\end{pmatrix}$$

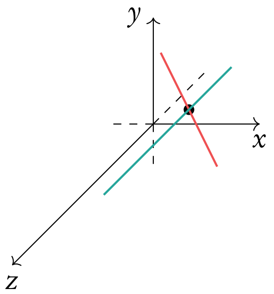
$\Rightarrow -2 = 1 - t \Rightarrow t = 3$

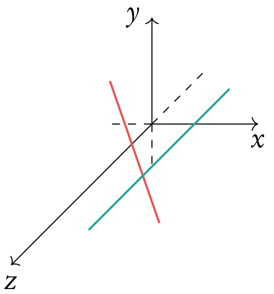
$\Rightarrow 9 = 3 - 4t \Rightarrow 9 \neq 3 - 12$

$\Rightarrow R$ does not lie on the line.

If one considers two lines in a three-dimensional graph, then there are three ways in which they can interact:


Parallel


Intersecting


Skew

If direction vectors defining a line aren't multiples of one another, then the lines can either be intersecting or skew. One can find out if the lines intersect by equating the vector equations and attempting to solve the set of equations (remember: one needs as many equations as variables to solve).

If one can't find a point of intersection, then the lines are skew.

Finding the intersection of two lines

Find the intersection for $r_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$ and $r_2 = \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$

1.	Equate write simultaneous equations.	$\begin{cases} 2 - 3s = -1 + 3t \\ 1 + s = 3 \end{cases}$
2.	Solve.	$s = 2,\ t = -1$
3.	Substitute back into r_1 or r_2.	$\begin{pmatrix} 2-3(2) \\ 1+2 \\ 4(2) \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix}$

4.1.3 Kinematics

Vectors can be used to describe the motion of objects. The relative position of object B from object A is the vector $\vec{AB}$. An objects position, $\mathbf{r}$, can be described using its initial position, $\mathbf{r}_0$, its velocity vector $\mathbf{v}$, and the time since its initial position, t.

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$

Example.

A particle begins at the position $\mathbf{r}_0$, and its velocity can be described by the vector $\mathbf{v}$. Find its position after 3 hours.

$$\mathbf{r}_0 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} t$$

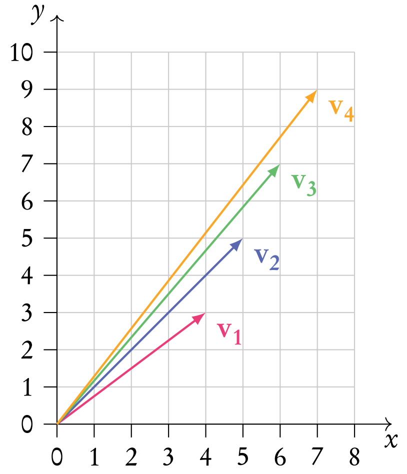
We can find the position using the formula, with $t = 3$.

$$\mathbf{r} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} 3 = \begin{bmatrix} 2 + 4 \times 3 \\ -3 + 2 \times 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \end{bmatrix}$$

An object can change velocity over time and vectors can be used to show this. A vector can have an additional variable, such as the time t, which allows the vector to change direction and magnitude over time. Filling in different values of t gives different vectors.

$$\mathbf{v_t} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 3 + t \\ 1 + 2t \end{bmatrix}$$

Variable vectors



4.2 Multiplying vectors

4.2.1 Dot (scalar) product

The dot product of two vectors $\vec{c}\cdot\vec{d}$ can be used to find the angle between them.

DB 4.2

$$\vec{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} \qquad \vec{d}=\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$$

$$\vec{c}\cdot\vec{d}=|\vec{c}||\vec{d}|\cos\vartheta$$

$$\vec{c}\cdot\vec{d}=c_1d_1+c_2d_2+c_3d_3$$

Often these two vectors are perpendicular.

Finding the angle between two lines

Find the angle between $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}8\\1\\3\end{pmatrix}$.

1.	Find the dot product in terms of components.	$\vec{c}\cdot\vec{d}=2\times8+3\times1+(-1)\times3=16$
2.	Find the dot product in terms of magnitudes.	$\vec{c}\cdot\vec{d}=\sqrt{2^2+3^2+(-1)^2}\times\sqrt{8^2+1^2+3^2}\times\cos\vartheta=\sqrt{14}\sqrt{74}\cos\vartheta$
3.	Equate and solve for ϑ.	$16=\sqrt{14}\sqrt{74}\cos\vartheta\Rightarrow\cos\vartheta=\frac{16}{\sqrt{14}\sqrt{74}}$ $\Rightarrow\vartheta=60.2^\circ$

When $\vartheta=90^\circ$ the vectors are perpendicular. As $\cos(90^\circ)=0\Rightarrow\vec{c}\cdot\vec{d}=0$ Learn to add the following statement to questions asking "*are they perpendicular?*".

$\vec{c}\cdot\vec{d}=0$ therefore $\cos x=0$, therefore $x=90^\circ$. Lines are perpendicular. Of course, when lines are not perpendicular replace all $=$ with $\neq$.

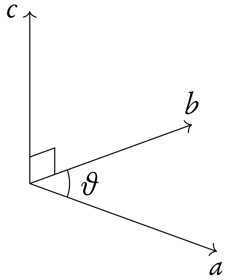
4.2.2 Cross (vector) product

The cross product of two vectors produces a third vector which is perpendicular to both of the two vectors. As the result is a vector, it is also called the *vector product*.

There are two methods to find the cross product:

1. $a \times b = |a||b| \sin \vartheta \mathrm{n}$
   where ϑ is the angle between a and b and n is a unit vector in the direction of c.
2. $x = a \times b$, where

$$c_1 = a_2 b_3 - a_3 b_2$$
$$c_2 = a_3 b_1 - a_1 b_3$$
$$c_3 = a_1 b_2 - a_2 b_1$$



Example.

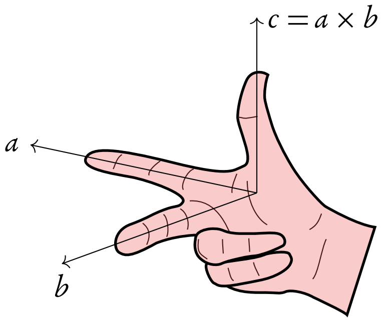
Find the cross product of $a \times b$.
$a = (2,3,4)$, $b = (5,6,7)$.

$$c_1 = 3 \times 7 - 4 \times 6 = -3$$
$$c_2 = 4 \times 5 - 2 \times 7 = 6$$
$$c_3 = 2 \times 6 - 3 \times 5 = -3$$

$\Rightarrow a \times b = (-3,6,-3)$

Remember the cross product is not commutative, so $a \times b \neq b \times a$.

You can check the direction of c with the right hand rule:



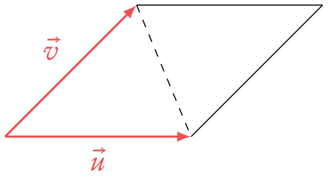
4.2.3 Geometric interpretation of vector product

The length of the cross product can be found by either of the two methods:

1. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\vartheta)$ where ϑ is the angle between vectors $\vec{u}$ and $\vec{v}$.
2. By calculating the vector with use of cross product formula and then finding the length of that vector.

However, there are two main interpretations of length of the vector product:

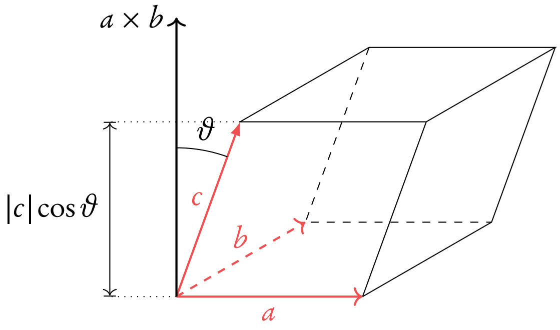
1. The length of the vector, that you get from cross product of two vectors.
2. Area of a parallelogram made up from the two vectors.



It also means, that half of the length of the cross product will be the area of triangle made up from the two original vectors.
However, the vector product can also be used when finding the volume of parallelepiped, that is made up from three vectors. Usually, to find its volume, we need base × height. Base can be found with use of the vector product. Thus we get the following formula:

The volume of parallelepiped spanned with use of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is:

$$\left|(\vec{a} \times \vec{b}) \cdot \vec{c}\right|$$



Example.

Find the area of triangle with sides $\vec{a} = (1, 3, 5)$ and $\vec{b} = (-1, 2, 3)$.

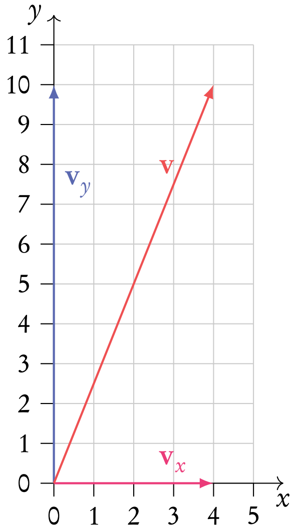
$$c_1 = 3 \times 3 - 5 \times 2 = -1$$
$$c_2 = 5 \times -1 - 1 \times 3 = -8$$
$$c_3 = 1 \times 2 - 3 \times -1 = 5$$
$$\vec{a} \times \vec{b} = (-1, -8, 5)$$
$$|\vec{a} \times \vec{b}| = |(-1, -8, 5)| = \sqrt{1 + 64 + 25} = 3\sqrt{10}$$

Thus the area of triangle is: $1.5\sqrt{10}$

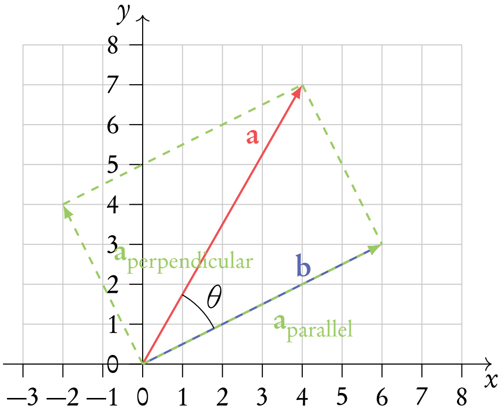
4.2.4 Vector components

Vectors can be broken down into their components.

For example, the vector $\mathbf{v} = 4\mathbf{i} + 10\mathbf{j}$ has a component of size 4, $\mathbf{v}_x$, acting in the x direction and a component of size 10, $\mathbf{v}_y$, acting in the y direction.



Components of vectors do not necessarily have to be in the x, y, or z directions though. We can work out the component of a vector **a** acting in the same direction as another vector **b**. We can also work out the component of **a** acting perpendicular to **b**, in the plane formed by the two vectors.



$\dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert} = \lvert\mathbf{a}\rvert\cos\theta$	component of vector **a** acting in the direction of vector **b**
$\dfrac{\mathbf{a}\times\mathbf{b}}{\lvert\mathbf{b}\rvert} = \lvert\mathbf{a}\rvert\sin\theta$	component of vector **a** acting perpendicular to vector **b**

Finding vector components

Find the component of **a** which acts parallel to **b**.

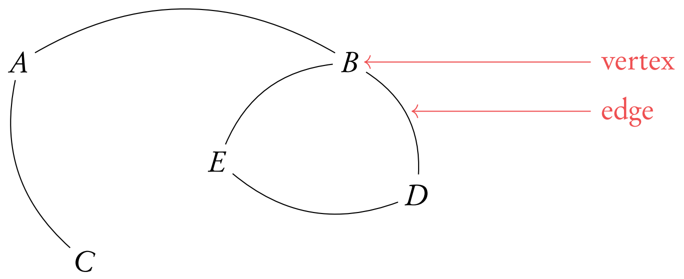
$$\mathbf{a} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

1.	Calculate the dot product between the two vectors.	$\mathbf{a}\cdot\mathbf{b} = 3\times(-1) + 7\times 4 = -3 + 28 = 25$
2.	Calculate the magnitude of **b**.	$\lvert\mathbf{b}\rvert = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$
3.	Fill these into the equation for the parallel component of the vector.	$\dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert} = \dfrac{25}{\sqrt{17}} = 6.0633\ldots$

4.3 Graph theory

4.3.1 Introduction

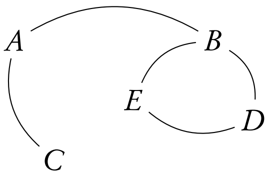
Graphs, sometimes called networks, are used to show the connections between different objects. They may show, for example, the roads between different towns, or the relationships between a group of people. Graphs are comprised of vertices and edges.



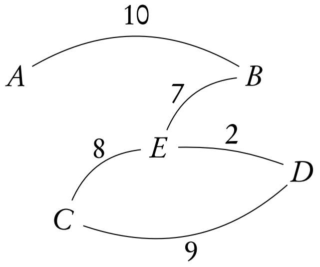
Adjacent vertices	two vertices that have an edge directly connecting them.
Adjacent edges	two edges which have a vertex in common.
Degree of a vertex	number of edges connected to it.

Example.

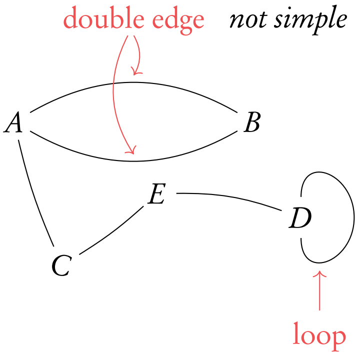
To find the adjacent vertices to vertex A we can follow all edges connected to A. The vertices they lead us to will be adjacent to A. Vertices B and C are adjacent to A. To find the adjacent edges to the edge connecting E and D (ED) we can look at the edges which leave vertices E and D. Edges EB and DB are adjacent to edge ED. To find the degree of vertex B we have to count all the edges connected to it. The degree of vertex B is 3.



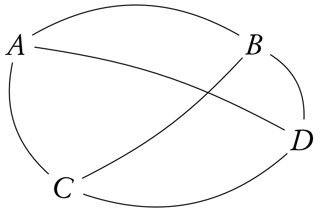
4.3.2 Types of graphs



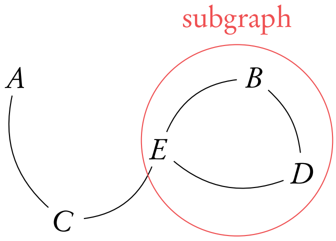
Weighted graphs assign a value to each edge.
The weights assigned to each edge may represent the time it takes to walk between two cities, distances, or costs.



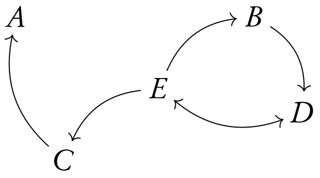
A **simple graph** is one which contains only one edge between any two vertices, and no loops.



A **complete graph** is one in which every pair of vertices is connected with one edge.



A **subgraph** is a graph which is contained within another graph.



In **directed graphs** each edge has a direction.
An edge from A to C is different to an edge from C to A. The direction of the edges are shown using arrows. Directed graphs can be used to show a network of social media accounts, for example. Then an arrow from one vertex to another would indicate that the first person is following the other person.

In degree of a vertex the number of incoming edges to the vertex.

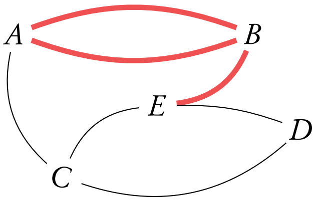
Out degree of a vertex the number of outgoing edges from the vertex.

4.3.3 Routes

Routes are ways in which we can walk around a graph. There are different types of routes. You need to be aware of these. A walk is any route taken though the graph. A walk begins at a vertex and follows edges to other vertices. It ends at a vertex and the length of a walk is equal to the number of edges it moves along. Vertices and edges can be repeated in walks.

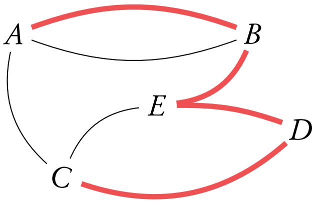
A trail is a walk with no repeated edge.

Trail
from B to A to B to E



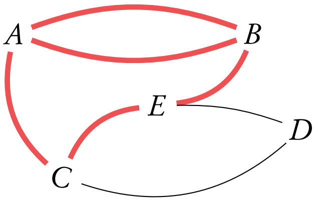
A path is a walk with no repeated vertex.

Path
from C to D to E to B to A



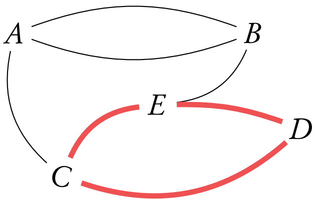
A circuit is a closed trail.
No edges are repeated and the trail begins and ends at the same vertex.

Circuit
from B to A to B to E to C to A

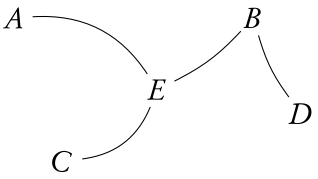


A cycle is a circuit which has no repeated vertices, apart from the beginning/end vertex.

Cycle
from D to E to C to D



A tree is an undirected graph in which any two vertices are connected by exactly one path.
Trees contain no cycles.

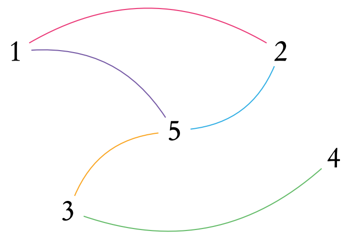


4.3.4 Adjacency matrices

Adjacency matrices, $\mathbf{A}$, can be used to describe graphs. They store information about the edges and paths between different vertices of a graph. An adjacency matrix is always a square matrix with the same number of rows/columns as vertices in the graph it represents. The rows and columns of adjacency matrices represent the vertices in the graph. An entry in the 4th row and 5th column of an adjacency matrix tells us something about the vertex between the 4th and 5th vertex.

For unweighted graphs we fill in a 0 into the adjacency matrix to show there are no edges connecting the two vertices. We fill in a 1 if an edge connects the two vertices.

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$



The $(i,j)^{\text{th}}$ entry of $\mathbf{A}^{\mathrm{k}}$ gives the number of k-length walks between vertex i and vertex j.

Using matrices to find walks in graphs

Find the number of walks which are of length 3 or less between vertex 2 and vertex 3 in the graph described by matrix $\mathbf{A}$.

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

1. Input the adjacency matrix into your calculator.

2. Raise the matrix to the power of the length of walks you are looking for.

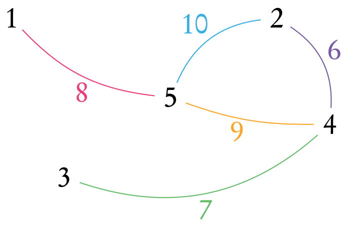
In this case we want to know the number of 3-length, 2-length and 1-length walks.

$$\mathbf{A}^2=\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^3=\begin{bmatrix} 2 & 4 & 3 & 1 \\ 4 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

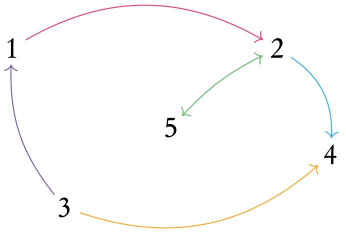
3.	Look for the entry in your matrix or matrices which correspond to the mentioned vertices.	In our case we are looking at entry $a_{3,2}$ or $a_{2,3}$. $\mathbf{A}$: $a_{3,2} = 1$ $\mathbf{A}^2$: $a_{3,2} = 1$ $\mathbf{A}^3$: $a_{3,2} = 4$ This tells us that between vertices 2 and 3 there are: 4 walks of length 3, 1 walk of length 2, and 1 walk of length 1. In total this is 6 walks of length 3 or less.

Adjacency matrices can also be created for weighted and directed graphs. Instead of inputting 1s and 0s into the matrix, for weighted graphs we instead enter the weights of the edges. If there is no edge we input a zero. For directed graphs the rows represent the starting vertex and the columns the end vertex. When we input the $(i, j)^{\text{th}}$ element we are looking at edges *from* vertex i *to* vertex j.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 6 & 10 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 6 & 7 & 0 & 9 \\ 8 & 10 & 0 & 9 & 0 \end{bmatrix}$$



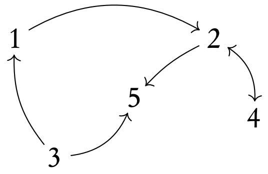
$$\mathbf{A} = \begin{array}{cc} & \longrightarrow \text{to} \\ \text{from}\uparrow & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



4.3.5 Transition matrices

A transition matrix looks similar to an adjacency matrix. If there is no edge from vertex i to vertex j then a zero is entered for the $(i, j)^{\text{th}}$ entry. If there is an edge *from* vertex j *to* vertex i then 1 divided by the (out) degree of vertex j is input into the matrix. Note that in these matrices, we move from the column number vertex to the row number vertex. These can be constructed for both directed and undirected graphs.

Example.



To find the transition matrix of this graph, we first write down the out-degrees of each vertex, d_i.

$$\begin{array}{ccccc} d_1=1 & d_2=2 & d_3=2 & d_4=1 & d_5=0 \\ \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \frac{1}{d_1}=1 & \frac{1}{d_2}=\frac{1}{2} & \frac{1}{d_3}=\frac{1}{2} & \frac{1}{d_4}=1 & \frac{1}{d_5}=0 \end{array}$$

The we begin to fill in the transition matrix, beginning with the first column. This represents the edges running from vertex 1 to all the other vertices. There is only one edge leaving vertex 1. It goes to vertex 2. The $(2, 1)^{\text{th}}$ entry in our matrix will therefore be $\frac{1}{d_1} = 1$. All other $(i, 1)^{\text{th}}$ entries are zero as no other edges leave vertex 1. Each column in a transition matrix adds up to 1.

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

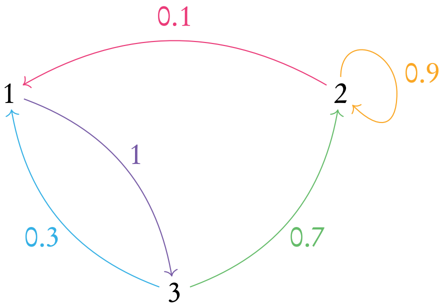
Transition matrices represent the probability of moving to each vertex, given the vertex you are currently on. The $(i, j)^{\text{th}}$ entry tells you the probability of moving to vertex i given you are in vertex j. We can use something called a column state matrix, or vector, $\mathbf{s}_n$, to show the probability of being in each vertex after n transitions, or moves between vertices. The initial probability of being in each of the vertices is given by $\mathbf{s}_0$. The probability of being in each of the vertices after n transitions is given by the following.

$$\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$$

4.3.6 Markov chains

Markov chains are directed, weighted graphs in which the weights of the edges represent the probability of traversing that edge. Transition matrices can be used to represent them. Remember transition matrices work from the column to the row, unlike adjacency matrices.

$$\mathbf{T} = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 0 & 0.9 & 0.7 \\ 1 & 0 & 0 \end{bmatrix}$$



Example.

To find the probability of being in each vertex after 10 transitions, given we started in vertex 1, we need to set up an initial state matrix, $\mathbf{s}_0$.

$$\mathbf{s}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now we can use the following formula along with our transition matrix to give the probabilities of being in each vertex after 10 transitions. Your calculator can work this out for you.

$$\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$$

$$\mathbf{s}_{10} = \mathbf{T}^{10} \mathbf{s}_0$$

$$\mathbf{s}_{10} = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 0 & 0.9 & 0.7 \\ 1 & 0 & 0 \end{bmatrix}^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.112 \\ 0.778 \\ 0.110 \end{bmatrix}$$

Now we have a column state matrix, $\mathbf{s}_{10}$ which tells us the probability of being in each vertex after 10 transitions given we started in vertex 1.

When working with transition matrices, after many transitions, the probability that we are in each vertex will become stable and constant. This is known as the steady state. One way the steady state can be found is by looking at the column state matrix after many transitions. For this the initial state does not matter, as so many transitions are applied to it.

$$\mathbf{s}_{100} = \begin{bmatrix} 0 & 0.1 & 0.3 \\ 0 & 0.9 & 0.7 \\ 1 & 0 & 0 \end{bmatrix}^{100} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.111 \\ 0.777 \\ 0.111 \end{bmatrix}$$

The column state matrix reaches a point at which it stops changing after being put through the transition matrix. This is the steady state vector. The steady state vector's components always add to 1. Another way to find the the steady state is by using the eigenvectors and eigenvalues of the transition matrix.

The steady state vector is equal to the eigenvector of the transition matrix with an eigenvalue of 1.

Finding the steady state vector

Find the steady state vector of the following transition matrix.

$$\mathbf{T} = \begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix}$$

1. Find the eigenvalues first, using the characteristic equation.

$$\det\left(\begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 0.5-\lambda & 0.2 \\ 0.5 & 0.8-\lambda \end{bmatrix}\right) = 0$$

$$(0.5-\lambda)\times(0.8-\lambda)-(0.2\times 0.5) = 0$$

$$0.4-1.3\lambda+\lambda^2-0.1 = 0$$

$$\lambda^2-1.3\lambda+0.3 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 0.3$$

2. Use the eigenvalue equal to 1 to find its corresponding eigenvector.

$$\left(\begin{bmatrix} 0.5 & 0.2 \\ 0.5 & 0.8 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\right)\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

$$\begin{bmatrix} -0.5 & 0.2 \\ 0.5 & -0.2 \end{bmatrix}\begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

$$-0.5v_{1,1}+0.2v_{1,2} = 0$$

$$0.5v_{1,1}-0.2v_{1,2} = 0$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$$

3. Scale the vector so that its components sum to 1.

We can do this by dividing each component by the sum of all components.

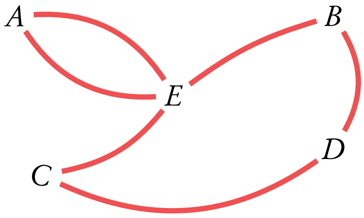
$$\mathbf{v}_1 = \begin{bmatrix} \dfrac{0.4}{0.4+1} \\ \dfrac{1}{0.4+1} \end{bmatrix} = \begin{bmatrix} 0.286 \\ 0.714 \end{bmatrix}$$

4.4 Algorithms

4.4.1 Eulerian trails and Hamiltonian paths

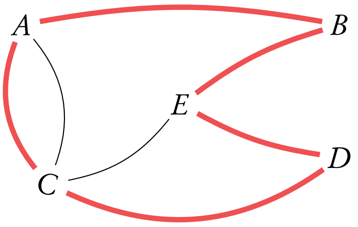
A trail or circuit is **Eulerian** if it crosses each edge only once, and ends up in the same vertex that it began in.
A graph has an Eulerian circuit if and only if every vertex has an even degree.
A graph has an Eulerian trail if and only if there are at most two vertices with an odd degree.

Eulerian circuit
from A to E to C
to D to B to E to A



A path or cycle is **Hamiltonian** if it reaches each vertex only once, and ends up in the same vertex it began in.

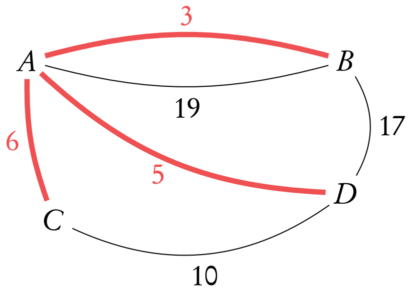
Hamiltonian circuit
from A to B to E to D to C to A



4.4.2 Minimum spanning trees

A **minimum spanning tree** (MST) is a subgraph which connects all vertices in the main graph using the minimum required edges. In weighted graphs the MST has the smallest total edge weight whilst connecting all vertices.

Minimum spanning trees



There are two algorithms for finding the MST which you need to know: *Kruskal*'s and *Prim*'s.

Prim's algorithm

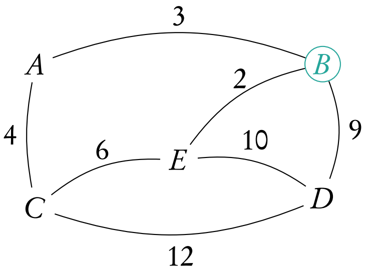
1. Begin the tree with a vertex in the graph. This can be any vertex.
2. Look for edges which connect the tree to vertices which are not yet in the tree. Choose the smallest weighted edge of these to add to the tree.
3. Repeat step 2 until all vertices are in the tree.

Example.

Prim's algorithm

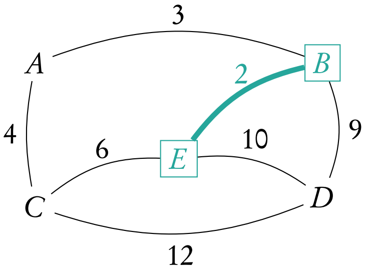
Step 1: We pick vertex B to begin in.
We must choose the smallest weighted edge connected to it.
The options are highlighted with a circle.

$$\mathbf{A} = \begin{bmatrix} 0 & \textcircled{3} & 4 & 0 & 0 \\ \textcircled{3} & \textcircled{0} & \textcircled{0} & \textcircled{9} & \textcircled{2} \\ 4 & \textcircled{0} & 0 & 12 & 6 \\ 0 & \textcircled{9} & 12 & 0 & 10 \\ 0 & \textcircled{2} & 6 & 10 & 0 \end{bmatrix}$$



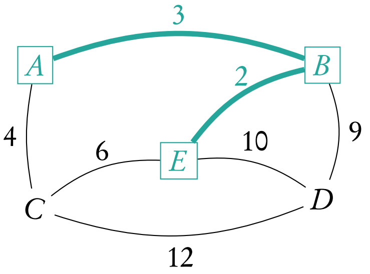
Step 2: We pick the edge between E and B, highlighted with a rectangle.
Now we can choose between any edge connecting E or B to the other vertices.

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 4 & 0 & 0 \\ 3 & 0 & 0 & 9 & \boxed{2} \\ 4 & 0 & 0 & 12 & 6 \\ 0 & 9 & 12 & 0 & 10 \\ 0 & \boxed{2} & 6 & 10 & 0 \end{bmatrix}$$



Step 3: We pick the edge between A and B.
Now we can choose between any edge connecting A or B or E to the other vertices.

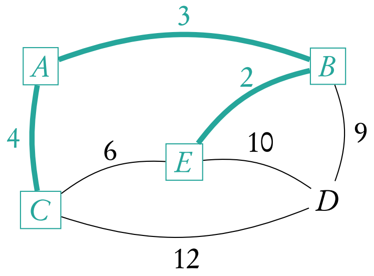
$$\mathbf{A} = \begin{bmatrix} 0 & \boxed{3} & 4 & 0 & 0 \\ \boxed{3} & 0 & 0 & 9 & \boxed{2} \\ 4 & 0 & 0 & 12 & 6 \\ 0 & 9 & 12 & 0 & 10 \\ 0 & \boxed{2} & 6 & 10 & 0 \end{bmatrix}$$



Step 4: We pick the edge between A and C. Now we can choose between any edge connecting A or B or C or E to the remaining vertex D. We must make sure to pick an edge in the 4th row/column, which corresponds to vertex D.

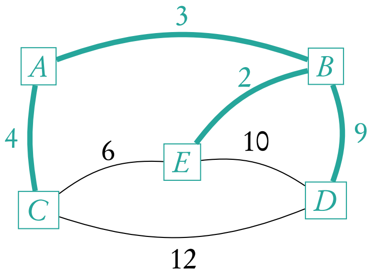
Example.

$$\mathbf{A} = \begin{bmatrix} 0 & \boxed{3} & \boxed{4} & 0 & 0 \\ \boxed{3} & 0 & 0 & 9 & \boxed{2} \\ \boxed{4} & 0 & 0 & 12 & 6 \\ 0 & 9 & 12 & 0 & 10 \\ 0 & \boxed{2} & 6 & 10 & 0 \end{bmatrix}$$



Step 5: We pick the vertex between B and D. Now every vertex in included in our MST.
The edges highlighted with a rectangle are the edges which make up the tree.

$$\mathbf{A} = \begin{bmatrix} 0 & \boxed{3} & \boxed{4} & 0 & 0 \\ \boxed{3} & 0 & 0 & \boxed{9} & \boxed{2} \\ \boxed{4} & 0 & 0 & 12 & 6 \\ 0 & \boxed{9} & 12 & 0 & 10 \\ 0 & \boxed{2} & 6 & 10 & 0 \end{bmatrix}$$



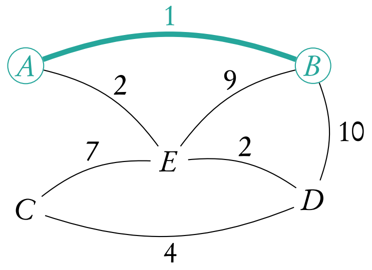
Kruskal's algorithm

1. Find the smallest weighted edge in the graph.
2. Find the next smallest weighted edge in the graph, provided that it does not form a cycle. This edge does not have to be adjacent to the existing tree edges.
3. Repeat step 2 until all vertices are included in the tree.

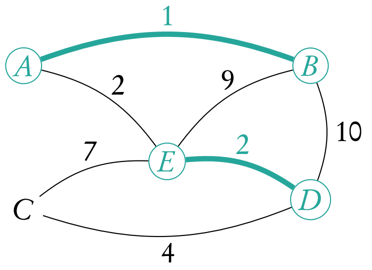
Example.

Kruskal's algorithm

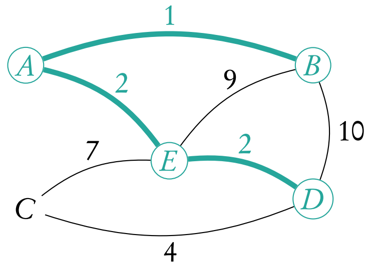
Step 1:



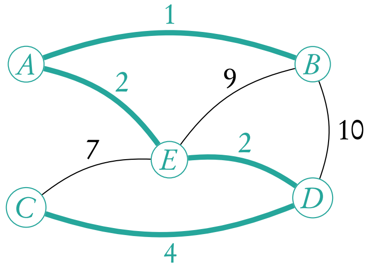
Step 2:



Step 3:



Step 4:



4.4.3 The Chinese postman problem

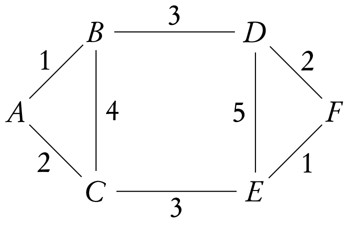
The Chinese postman problem is all about finding the shortest route around a weighted graph which goes along each edge at least once, starting and finishing at the same vertex. If the graph has an Eulerian circuit, then this is the solution to the Chinese postman problem – every edge has been crossed exactly once. If the graph does not have an Eulerian circuit then an algorithm must be applied.

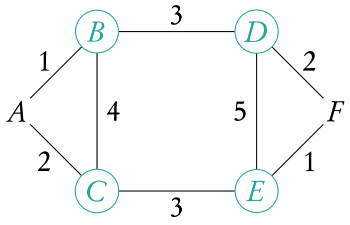
1. Find all the odd vertices in the graph (vertices with an odd degree).
2. Write down all possible pairings of these vertices.
3. For each pair, write down the length of the shortest possible path between them. Note which edges you take.
4. Determine the combination of pairings which has the shortest total length.
5. For this pairing of odd vertices, draw on extra edges to the graph alongside the edges you took to connect the vertices in step 3.
6. Now find a route which crosses every edge with the shortest distance. The extra edges you drew onto the graph must also be used.

Example.

The Chinese postman problem

Find the shortest route around the graph which begins and ends in vertex A.





Step 1: Identify the odd vertices. They are B, C, D, and E.

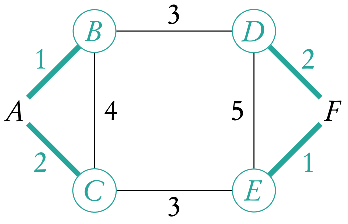
Step 2: Write out all the possible pairing of these vertices.

BC and DE
BD and CE
BE and DC

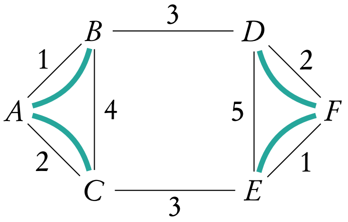
Example.

Step 3: Write down the lengths of the shortest paths between each pair.
The shortest path between B and C is via A. It has length 3.

$$BC = 3 \text{ and } DE = 3$$
$$BD = 3 \text{ and } CE = 3$$
$$BE = 6 \text{ and } DC = 6$$



Step 4: The set of pairs with the shortest total length is either BC and DE or BD and CE. We can choose either to continue. For this example we will choose BC and DE. Now we add extra edges to the graph along the shortest route we took from B to C and from D to E.



Step 5: We can now begin to search for the shortest route using our new extra edges. We begin in A and move to B then to D. We take D to F and then go back to D using the extra edge. Now we go to E and now we must go to F so that these edges are used. We move to F and then back to E then to C and B and A. The final edges are crossed by moving from A to C back to A.
The route is $ABDFDEFECBACA$.

4.4.4 The travelling salesman problem

The travelling salesman problem is about finding a circuit of least weight in a weighted graph. Each vertex must be visited at least once. There are no algorithms to find this circuit, but there are algorithms to determine the upper and lower bounds for the length of the circuit.

The upper bound uses the nearest neighbour algorithm. Starting from different vertices may give different upper bounds.

1. Choose a vertex to start in.
2. Move to the closest neighbouring vertex, taking into account the weights of the edges.
3. Repeat step 2 until you reach the last vertex.
4. Find the shortest route to move from this last vertex back to the vertex you began in from step 1.
5. Add up the weights of the edges you have crossed. This is the upper bound for the travelling salesman problem for your graph.

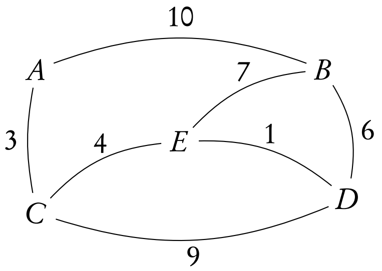
The lower bound uses the deleted vertex algorithm. Deleting different vertices may give different lower bounds.

1. Choose a vertex to delete. Remove this vertex from the graph, along with any edges connected to it.
2. Find the length of the MST for the remaining vertices. Use Prim's or Kruskal's algorithm.
3. Return the deleted vertex to the graph. Connect it to the MST using the shortest two edges.
4. Add the lengths of these edges to the length of the MST. This is your lower bound for the travelling salesman problem for your graph.

Example.

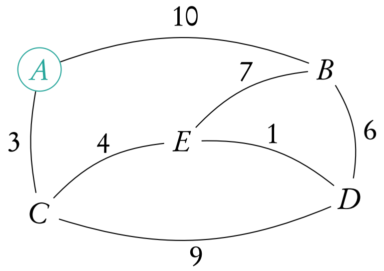
The travelling salesman problem

Find an upper and lower bound to the travelling salesman problem for the following graph..

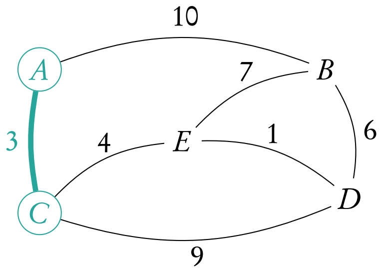


To find the upper bound we follow the nearest neighbour algorithm.

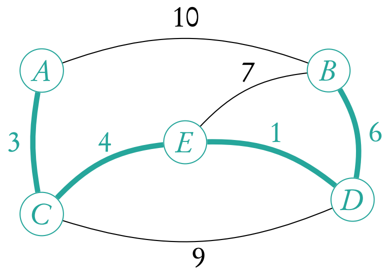
Step 1:



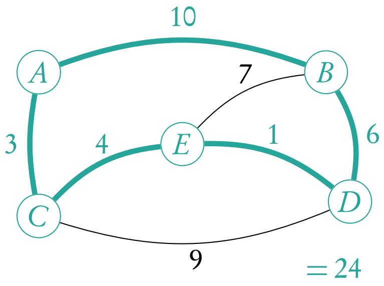
Step 2:



Step 3:



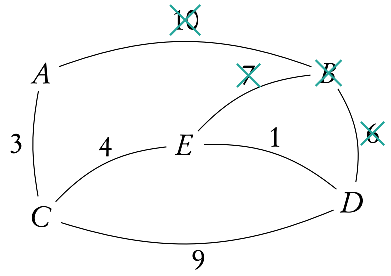
Step 4:



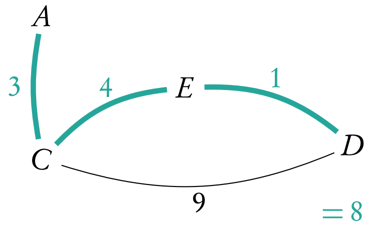
To find the lower bound we use the deleted vertex algorithm.

Example.

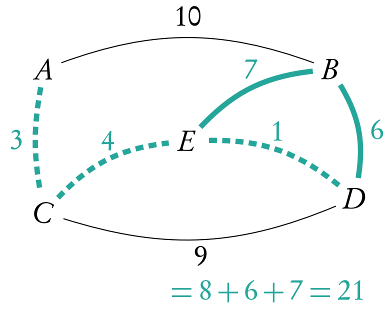
Step 1:



Step 2:



Step 3 and 4:



The lower bound is 21 and the upper bound is 24.

DIFFERENTIATION

5

Table of contents & cheatsheet

Definitions

Differentiation is a way to find the gradient of a function at any point, written as $f'(x)$, y' and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Tangent line to a point on a curve is a linear line with the same gradient as that point on the curve.

5.1.3. Tangent and normal 94

Tangent line with the same gradient as a point on a curve.

Normal perpendicular to the tangent $m = \dfrac{-1}{\text{slope of tangent}}$

Both are linear lines with general formula: $y = mx + c$.

1. Use derivative to find gradient of the tangent. For normal then do $-\dfrac{1}{\text{slope of tangent}}$.
2. Input the x-value of the point into $f(x)$ to find y.
3. Input y, m and the x-value into $y = mx + c$ to find c.

Sketching graphs

Gather information before sketching:

Intercepts x-intercept: $f(x) = 0$
y-intercept: $f(0)$

Turning points minima: $f'(x) = 0$ and $f''(x) < 0$
maxima: $f'(x) = 0$ and $f''(x) > 0$
point of inflection: $f''(x) = 0$

Asymptotes vertical: x-value when the function divides by 0
horizontal: y-value when $x \to \infty$

Plug the found x-values into $f(x)$ to determine the y-values.

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Product $y = uv$, then: $y' = uv' + u'v$

Quotient $y = \dfrac{u}{v}$, then: $y' = \dfrac{vu' - uv'}{v^2}$

Chain $y = g(u)$ where $u = f(x)$, then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

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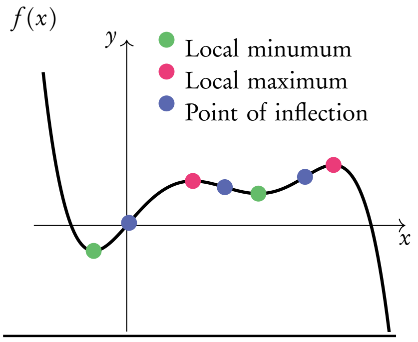


	$f'(x)$	$f''(x)$
Local minimum	0	+
Local maximum	0	−
Points of inflection		0

5.2.2. Applications 100

Kinematics

Derivative represents the rate of change, integration the reverse.

$$\begin{tikzcd} s \arrow[r, bend left, "\frac{\mathrm{d}s}{\mathrm{d}t} = v"] & v \arrow[r, bend left, "\frac{\mathrm{d}v}{\mathrm{d}t} = a"] & a \\ \text{displacement} & \text{velocity} & \text{acceleration} \end{tikzcd}$$

5.1 Introduction

As you have learnt in the unit on functions, a straight line graph has a gradient. This gradient describes the rate at which the graph is changing and using it we can tell how steep the line will be when plotted on a graph. In fact, gradients can be found for any function; the special thing about linear functions is that their gradient is always the same (given by m in $y = mx + c$).

Non-linear functions however, will have changing gradients. Their steepness will be different at different x-values. This is where calculus comes in handy; we can use differentiation to derive a function using which we can find the gradient for any value of x. Two types of notation are used for calculus.

Function	Gradient Function
$f(x)$	$f'(x)$
y	$\frac{dy}{dx}$

5.1.1 Polynomials

As functions forming curved lines, the gradients of polynomials are changing at each point. You can find the derivative function ($f'(x)$) for any polynomial function ($f(x)$) using the principles explained below.

Polynomial a function that contains one or more terms often raised to different powers

e.g. $y = 3x^2$, $y = 121x^5 + 7x^3 + x$ or $y = 4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}$

DB 5.3

Principles $y = f(x) = ax^n \Rightarrow \frac{dy}{dx} = f'(x) = nax^{n-1}$

the (original) function is described by y or $f(x)$

the derivative (gradient) function is described by $\frac{dy}{dx}$ or $f'(x)$

Derivative of a constant (number) 0

e.g. for $f(x) = 5$, $f'(x) = 0$

Derivative of a sum sum of derivatives.

If a function you want to differentiate is made up of several summed parts, find the derivatives for each part separately and then add them together again.

e.g. $f(x) = ax^n$ and $g(x) = bx^m$

$$f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$$

When differentiating it is useful to first rewrite the polynomial function into a form that is easy to differentiate. Practically this means that you may need to use the laws of exponents before (or after) differentiation to simplify the function.

For example, $y = \dfrac{5}{x^3}$ seems difficult to differentiate, but using the laws of exponents we know that $y = \dfrac{5}{x^3} = 5x^{-3}$. Having the equation in this form allows you to apply the same principles as you would use to differentiate any other polynomial.

Example.

$f(x)$		$f'(x)$
5	$\longrightarrow$	0
x^2	$\longrightarrow$	$2 \cdot 1x^{2-1} = 2x$
$4x^3$	$\longrightarrow$	$3 \cdot 4x^{3-1} = 12x^2$
$3x^5 - 2x^2$	$\longrightarrow$	$5 \cdot 3x^{5-1} - 2 \cdot 2x^{2-1} = 15x^4 - 4x$
$\dfrac{2}{x^4} = 2x^{-4}$	$\longrightarrow$	$(-4) \cdot 2x^{-4-1} = -8x^{-5} = \dfrac{-8}{x^5}$
$3x^4 - \dfrac{2}{x^3} + 3$	$\longrightarrow$	$4 \cdot 3x^{4-1} - 3 \cdot (-2)x^{-3-1} + 0 = 12x^3 + \dfrac{6}{x^4}$

5.1.2 Rules

With more complicated functions, in which several functions are being multiplied or divided by one another (rather than just added or subtracted), you will need to use the product or quotient rules.

DB 5.6

Product rule

When functions are *multiplied*: $y = uv$

then: $y' = uv' + u'v$

which is the same as $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Quotient rule

When functions are *divided*: $y = \dfrac{u}{v}$

then: $y' = \dfrac{vu' - uv'}{v^2}$

which is the same as $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Example.

Let $y = x^2 \cos x$, then

$$y' = x^2(\cos x)' + \left(x^2\right)' \cos x$$

$$= -x^2 \sin x + 2x \cos x$$

Let $y = \dfrac{x^2}{\cos x}$, then

$$y' = \frac{\left(x^2\right)' \cos x - x^2(\cos x)'}{(\cos x)^2}$$

$$= \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$$

Chain rule

A function inside another function is a composite function, $f \circ g(x)$, which we discussed in the Functions chapter

When a function is inside another function: $y = g(u)$ where $u = f(x)$

then: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

Differentiating with the chain rule

Let $y = (\cos x)^2$, determine the derivative y'

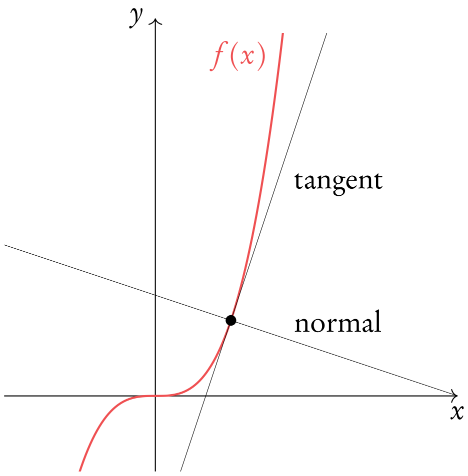
1.	Determine what the inside (u) and outside (y) functions are.	Inside function: $u = \cos x$ Outside function: $y = u^2$
2.	Find u' and y'.	$u' = \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x; \quad y' = \frac{\mathrm{d}y}{\mathrm{d}u} = 2u$
3.	Fill in chain rule formula.	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$ $= 2u(-\sin x)$ $= -2\sin x \cos x$

5.1.3 Tangent and normal equations

Tangent a straight line that touches a curve at one single point. At that point, the gradient of the curve is equal to the gradient of the tangent.

Normal a straight line that is perpendicular to the tangent line:

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}}$$



Finding the linear function of the tangent

Let $f(x)=x^3$. Find the equation of the tangent at $x=2$

1.	Find the derivative and fill in value of x to determine slope of tangent.	$f'(x)=3x^2$ $f'(2)=3\cdot 2^2=12$
2.	Determine the y value.	$f(x)=2^3=8$
3.	Plug the slope m and the y value in $y=mx+c$.	$8=12x+c$
4.	Fill in the value for x to find c.	$8=12(2)+c$ $c=-16$ eq. of tangent: $y=12x-16$

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

Finding the linear function of the normal

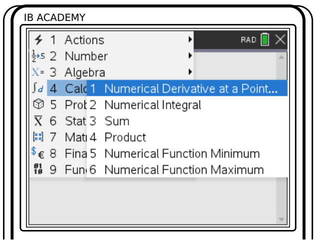
Let $f(x)=x^3$. Find the equation of the normal at $x=2$

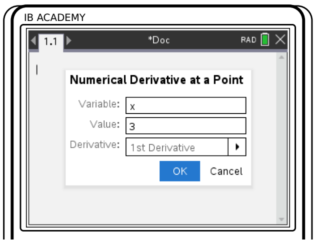
1.	____________________.	$f'(2)=12$
2.	____________________.	$f(x)=8$
3.	Determine the slope of the normal $m=\dfrac{-1}{\text{slope tangent}}$ and plug it and the y-value into $y=mx+c$.	$m=\dfrac{-1}{12}$ $8=-\dfrac{1}{12}x+c$
4.	Fill in the value for x to find c.	$8=-\dfrac{1}{12}(2)+c$ $c=\dfrac{49}{6}$ eq. of normal: $y=-\dfrac{1}{12}x+\dfrac{49}{6}$

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

To find the gradient of a function for any value of x.

$f(x) = 5x^3 - 2x^2 + x$. Find the gradient of $f(x)$ at $x = 3$.



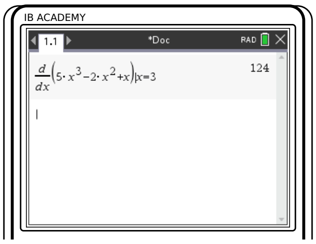


Numerical Derivative at a Point

Variable: x

Value: 3

Derivative: 1st Derivative



$\frac{d}{dx}(5\cdot x^3-2\cdot x^2+x)|x=3$ 124

Press menu
`4: Calculus`
`1: Numerical Derivative at a Point`

Enter the variable used in your function (x) and the value of x that you want to find. Keep the settings on `1st Derivative`
Press OK

Type in your function
press enter

In this case, $f'(3) = 124$

5.2 Turning points

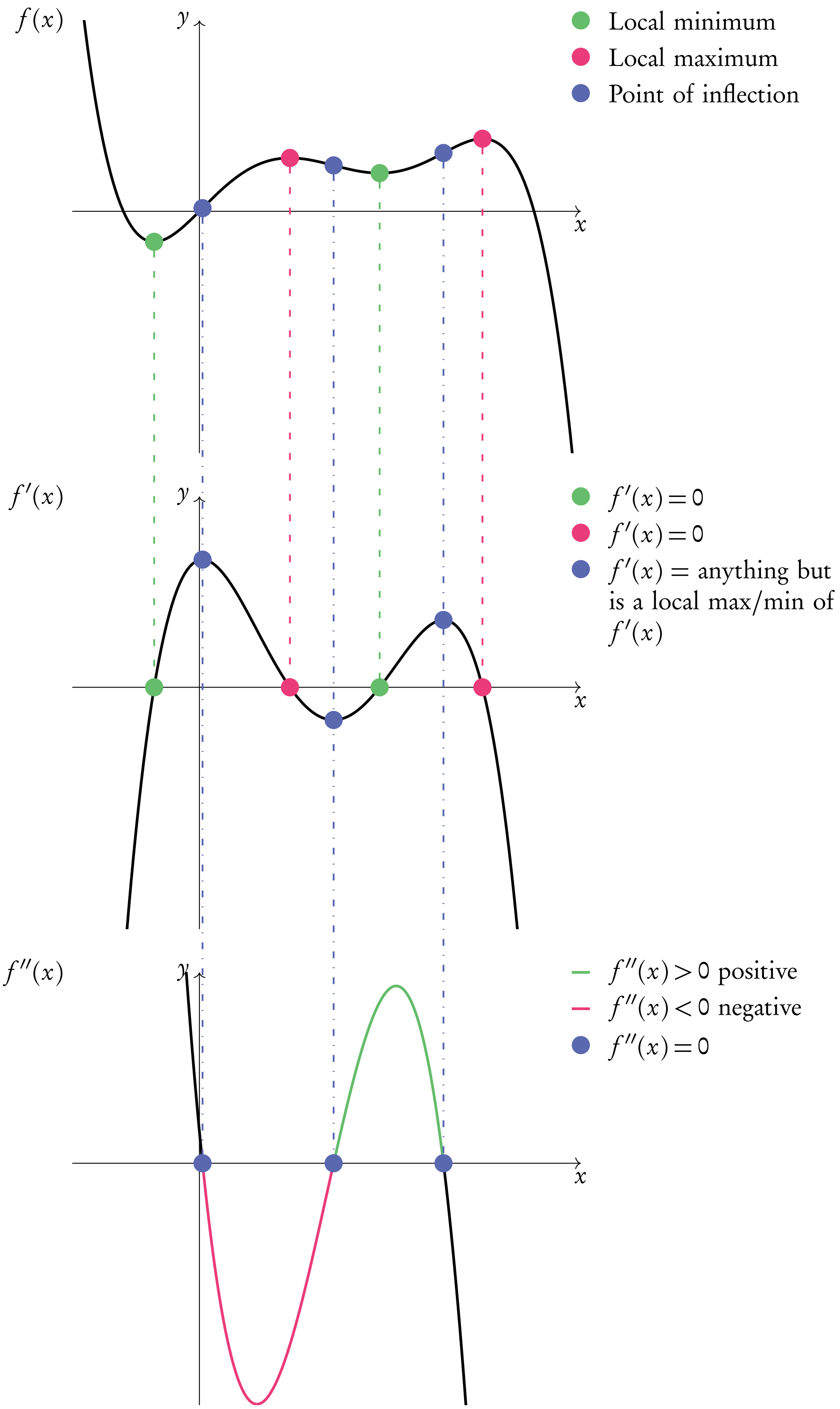
There are three types of turning points:

1. Local maxima
2. Local minima
3. Points of inflection

We know that when $f'(x) = 0$ there will be a maximum or a minimum. Whether it is a maximum or minimum should be evident from looking at the graph of the original function. If a graph is not available, we can find out by plugging in a slightly smaller and slightly larger value than the point in question into $f'(x)$. If the smaller value is negative and the larger value positive then it is a local minimum. If the smaller value is positive and the larger value negative then it is a local maximum.

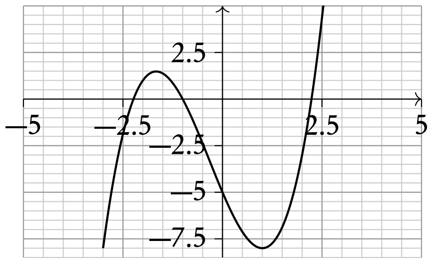
If you take the derivative of a derivative function (one you have already derived) you get the *second derivative*. In mathematical notation, the second derivative is written as y'', $f''(x)$ or $\frac{d^2y}{dx^2}$. We can use this to determine whether a point on a graph is a maximum, a minimum or a point of inflection as demonstrated in the following Figure 5.1.

Figure 5.1: Graph that shows a local maximum, a local minimum and points of inflection



Notice how the points of inflection of $f(x)$ are minima and maxima in $f'(x)$ and thus equal 0 in $f''(x)$

Finding turning points



The function $f(x)=x^3+x^2-5x-5$ is shown. Use the first and second derivative to find its turning points: the minima, maxima and points of inflection (POI).

1. Find the first and second derivative.

$$f'(x)=3x^2+2x-5$$
$$f''(x)=6x+2$$

2. Find x_{min} and x_{max} by setting $f'(x)=0$.

$$3x^2+2x-5=0$$

GDC yields: $x=1$ or $x=-\frac{5}{3}$

3. Find y-coordinates by inserting the x-value(s) into the original $f(x)$.

$f(1)=(1)^3+(1)^2-5(1)-5=-8$, so x_{min} at $(1,-8)$.

$$f\left(-\frac{5}{3}\right)=\left(-\frac{5}{3}\right)^3+\left(-\frac{5}{3}\right)^2-5\left(-\frac{5}{3}\right)-5=1.48(3\text{ s.f.}),$$

so x_{max} at $\left(-\frac{5}{3},1.48\right)$.

4. Find POI by setting $f''(x)=0$.

$$6x+2=0$$

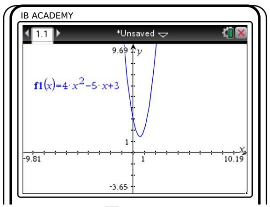
5. Enter x-values into original function to find coordinates.

$$f\left(-\frac{1}{3}\right)=\left(-\frac{1}{3}\right)^3+\left(-\frac{1}{3}\right)^2-5\left(-\frac{1}{3}\right)-5$$
$$y=-3.26\ (3\text{ s.f.})$$

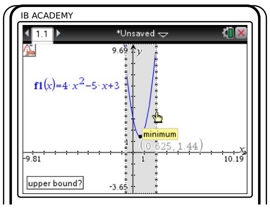
so POI at $\left(-\frac{1}{3},-3.26\right)$

Finding turning points (local maximum/minimum) of a function using GDC

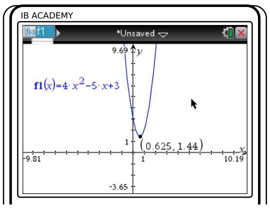
Find the coordinates of the local minimum for $f(x)=4x^2-5x+3$.



Press menu
`6: Analyze graph`
or `2: Minimum`
or `3: Maximum`



Use the cursor to set the bounds (the min/max must be between the bounds)



So the coordinates of the minimum for $f(x)$ are $(0.625,1.44)$

5.2.1 Optimisation

As we saw in the previous section, differentiation is useful for identifying maximum and minimum points of different functions. We can apply this knowledge to many real life problems in which we may seek to find maximum or minimum values; this is referred to as optimisation.

The most important thing to remember is that at a maximum or minimum point $f'(x) = 0$. So often if a question asks you to find a maximum/minimum value, just writing down $f'(x) = 0$ can score you points.

Determine the max/min value with certain constraints

The sum of the height h and base x of a triangle is 40 cm. Find an expression for the area in terms of x, hence find the maximum area of the triangle.

1. First write expression(s) for constraints followed by an expression for the actual calculation. Combine two expressions so that you are left with one variable.

$$\begin{aligned} x + h &= 40 \\ h &= 40 - x \\ A &= \frac{1}{2}xh \\ &= \frac{1}{2}x(40 - x) \\ &= -\frac{1}{2}x^2 + 20x \end{aligned}$$

2. Differentiate the expression.

$$\frac{\mathrm{d}A}{\mathrm{d}x} = -x + 20$$

3. The derivative $= 0$, solve for x.

$$\begin{aligned} -x + 20 &= 0 \\ x &= 20 \end{aligned}$$

4. Plug the x value into the original function.

$$\begin{aligned} A &= -\frac{1}{2}(20)^2 + 20(20) \\ &= -200 + 400 \\ &= 200\,\text{cm}^2 \end{aligned}$$

5.2.2 Kinematics

Kinematics deals with the movement of bodies over time. When you are given one function to calculate displacement, velocity or acceleration you can use differentiation or integration to determine the functions for the other two.

$$\begin{tikzcd} \text{Displacement, } s \arrow[d, bend right, "\frac{\mathrm{d}s}{\mathrm{d}t}"'] \\ \text{Velocity, } v = \frac{\mathrm{d}s}{\mathrm{d}t} \arrow[u, bend right, "\int v\,\mathrm{d}t"'] \arrow[d, bend right, "\frac{\mathrm{d}v}{\mathrm{d}t}"'] \\ \text{Acceleration, } a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2} \arrow[u, bend right, "\int a\,\mathrm{d}t"'] \end{tikzcd}$$

The derivative represents the rate of change, i.e. the gradient of a graph. So, velocity is the rate of change in displacement and acceleration is the rate of change in velocity.

In kinematics derivatives and second derivatives with respect to time, t, have a special notation.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \qquad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}$$

Answering kinematics questions

A diver jumps from a platform at time $t = 0$ seconds. The distance of the diver above water level at time t is given by $s(t) = -4.9t^2 + 4.9t + 10$, where s is in metres. Find when velocity equals zero. Hence find the maximum height of the diver.

1.	Differentiate or integrate to find required equation.	$v = s'$, so we differentiate the equation for $s(t)$: $v(t) = -9.8t + 4.9$
2.	Set equation equal to given value and solve.	$v(t) = 0$ $-9.8t + 4.9 = 0$ $t = 0.5$
3.	Plug solution back into the original function.	$s(0.5) = -4.9(0.5)^2 + 4.9(0.5) + 10 =$ 11.225 m

6

INTEGRATION AND DIFFERENTIAL EQUATIONS

Table of contents & cheatsheet

6.1.1. Indefinite integral 102

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

Integration with an internal function

$$\int f(ax+b)\,\mathrm{d}x$$

Integrate normally and multiply by $\dfrac{1}{\text{coefficient of } x}$

Integration by substitution

$$\int f\big(g(x)\big) \cdot g'(x)\,\mathrm{d}x$$

6.2. Definite integral 103

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,\mathrm{d}x$$

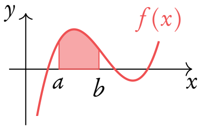
Be careful, the order you substitute a and b into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

Area between a curve and the x-axis

By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x$$

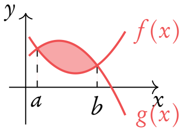


Note: the area below the x-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the x-axis. Sketching the graph will show what part of the function lies below the x-axis.

Area between two curves

Using definite integrals you can also find the areas enclosed between curves.

$$A_{\text{between}} = \int_a^b \big(g(x) - f(x)\big)\,\mathrm{d}x$$



With $g(x)$ as the "top" function (furthest from the x-axis). For the area between curves, it does not matter what is above/below the x-axis.

Volume of revolution

$$V = \pi \int_a^b y^2\,\mathrm{d}x = \int_a^b \pi y^2\,\mathrm{d}x$$

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis — this is called the volume of revolution.

6.1 Integration

6.1.1 Indefinite integrals and boundary conditions

Integration is essentially the opposite of differentiation; it can also be referred to as anti-differentiation. The following equation shows how to integrate a polynomial function:

DB 5.5

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

As you can see, every time you integrate the power on your variable will increase by 1; this is opposite of what happens with differentiation, when you subtract 1. Whenever you integrate you also **add** $+$**C** to this function. This accounts for any constant that may have been lost while differentiating.

Whenever you differentiate any constants that were in the original function, $f(x)$, become 0 in the derivative function, $f'(x)$.

In order to determine the value of C, you need to fill in a point that lies on the curve to set up an equation with which you can solve for C. This is called a boundary condition.

Finding indefinite integrals

Let $f'(x) = 12x^2 - 2$
Given that $f(-1) = 1$, find $f(x)$.

1.	Separate summed parts (optional).	$\int 12x^2 - 2\,\mathrm{d}x = \int 12x^2\,\mathrm{d}x + \int -2\,\mathrm{d}x$
2.	Integrate.	$f(x) = \int 12x^2\,\mathrm{d}x + \int -2\,\mathrm{d}x =$ $\frac{12}{3}x^3 - 2x + C$
3.	Fill in values of x and $f(x)$ to find C .	Since $f(-1) = 1$, $4(-1)^3 - 2(-1) + C = 1$ $C = 3$ So: $f(x) = 4x^3 - 2x + 3$

Note that this is the same thing you do when finding the y-intercept, c, for a linear function – see Functions: Linear models.

6.1.2 Integration by substitution

$$\int f\left(g(x)\right) \cdot g'(x)\,\mathrm{d}x$$

Integration by substitution questions are recognisable by a function and its derivative inside the function. Learning to spot these quickly is a matter of practice. Once you have identified the inside functions, the rest is fairly straight forward.

Integrate by substitution

Find $\int 3x^2 e^{x^3}\,\mathrm{d}x$

1.	Identify the inside function u, this is the function whose derivative is also inside $f(x)$.	$g(x) = u = x^3$
2.	Find the derivative $u' = \frac{\mathrm{d}u}{\mathrm{d}x}$.	$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$
3.	Substitute u and $\frac{\mathrm{d}u}{\mathrm{d}x}$ into the integral (this way $\mathrm{d}x$ cancels out).	$\int e^u \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u + C$
4.	Substitute u back to get a function with x.	$\int e^u + C = e^{x^3} + C$

6.2 Definite integral

If there are limit values indicated on your integral, you are looking to find a definite integral. This means that these values will be used to find a numeric answer rather than a function.

This is done in the following way, where the values for a and b are substituted as x-values into your indefinite integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,\mathrm{d}x$$

Be careful, the order you substitute a and b into the indefinite integral is relevant for your answer:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

Solving definite integrals

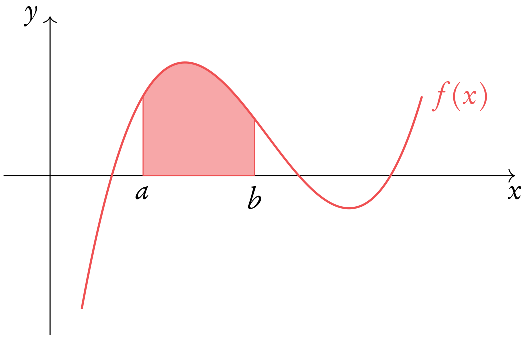
Find $\int_3^7 12x^2 - 2\,\mathrm{d}x$, knowing that $F(x) = 4x^3 - 2x$

1.	Find the indefinite integral (without $+C$).	$\int_3^7 12x^2 - 2\,\mathrm{d}x = \left[4x^3 - 2x\right]_3^7$
2.	Fill in: $F(b) - F(a)$ (integral $x = b$) $-$ (integral $x = a$).	$= \left[4(7)^3 - 2(7)\right] - \left[4(3)^3 - 2(3)\right]$ $= 1256$

6.2.1 Area

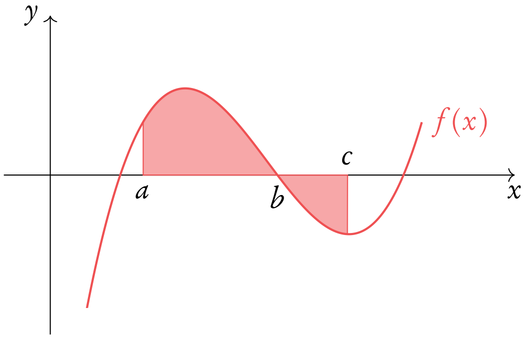
Area between a curve and the x-axis

DB 5.5



By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x$$



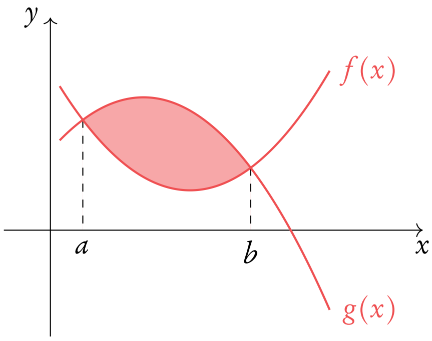
The area below the x-axis gives a negative value for its area. You must take that value as a positive value to determine the area between a curve and the x-axis. Sketching the graph will show what part of the function lies below the x-axis. So

$$A_{\text{curve}} = \int_a^b f(x)\,\mathrm{d}x + \left|\int_b^c f(x)\,\mathrm{d}x\right|$$

or

$$A_{\text{curve}} = \int_a^c |f(x)|\,\mathrm{d}x$$

Area between two curves



Using definite integrals you can also find the areas enclosed between curves:

$$A_{\text{between}} = \int_a^b \big(g(x) - f(x)\big)\,\mathrm{d}x$$

With $g(x)$ as the "top" function (furthest from the x-axis). For the area between curves, it does not matter what is above/below the x-axis.

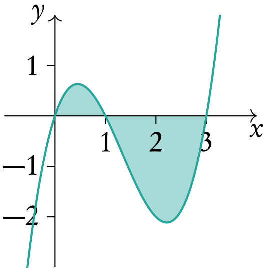
Finding areas with definite integrals

Let $y = x^3 - 4x^2 + 3x$
Find the area from $x = 0$ to $x = 3$.

1. Find the x-intercepts: $f(x) = 0$.

$x^3 - 4x^2 + 3x = 0$, using the GDC:
$x = 0$ or $x = 1$ or $x = 3$

2. If any of the x-intercepts lie within the range, sketch the function to see which parts lie above and below the x-axis.



3. Setup integrals and integrate.

Left: $\displaystyle\int_0^1 x^3 - 4x^2 + 3x \,dx =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) - (0)$$

$$= \frac{5}{12}$$

Right: $\displaystyle\int_1^3 x^3 - 4x^2 + 3x \,dx =$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_1^3$$

$$= \left(\frac{1}{4}(3)^4 - \frac{4}{3}(3)^3 + \frac{3}{2}(3)^2\right)$$

$$- \left(\frac{1}{4}(1)^4 - \frac{4}{3}(1)^3 + \frac{3}{2}(1)^2\right)$$

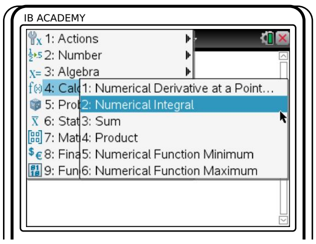
$$= -\frac{8}{3}$$

4. Add up the areas (and remember areas are never negative!)

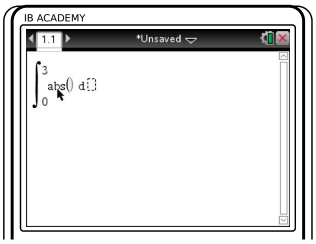
$$\frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

Alternatively, use the calculator to find areas

Calculate the area between $\int_0^3 x^3 - 4x^2 + 3x$ and the x-axis.

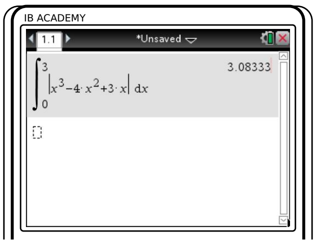


Press menu
`4: Calculus`
`2: Numerical integral`



$\int_0^3 \text{abs}()\,d$

Enter the boundaries. Before entering the function,
press 📖,
choose ‘`abs(`’



$\int_0^3 |x^3-4\cdot x^2+3\cdot x|\,dx$ 3.08333

Enter the function and place the variable (usually x) after d

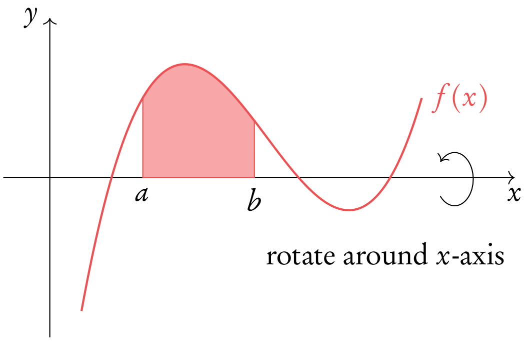
In this case, the area is 3.083

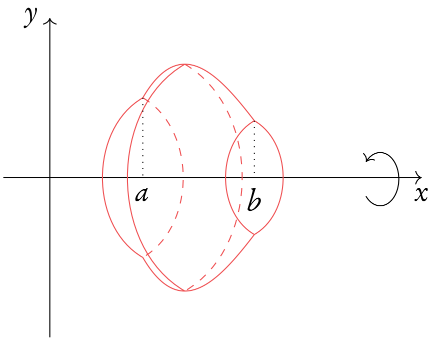
6.2.2 Volume of revolution

DB 6.5

Besides finding areas under and between curves, integration can also be used to calculate the volume of the solid that a curve would make if it were rotated 360° around its axis — this is called the volume of revolution.

$$V = \pi \int_a^b y^2\,\mathrm{d}x \quad \equiv \quad V = \int_a^b \pi y^2\,\mathrm{d}x$$





Example.

Find the area from $x = 1$ to $x = 4$ for the function $y = \sqrt{x}$.

$$A = \int_1^4 \sqrt{x}\,\mathrm{d}x = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3}(4)^{\frac{3}{2}}\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}}\right] = \frac{14}{3}$$

This area is rotated 360° $(= 2\pi)$ around the x-axis. Find the volume of the solid.

$$V = \pi\int_1^4 \sqrt{x}^2\,\mathrm{d}x = \pi\int_1^4 x\,\mathrm{d}x = \pi\left[\frac{1}{2}x^2\right]_1^4 = \pi\left(\left[\frac{1}{2}(4)^2\right] - \left[\frac{1}{2}(1)^2\right]\right) = \frac{15\pi}{2}$$

It is also possible to find the volume of revolution around the y-axis. It requires some additional steps, but in general it is not much different from the volume of revolution around the x-axis.

Find the volume of revolution

Find the volume of revolution of function $y = x^2$ from $x = 1$ to $x = 3$ around y-axis.

1.	Rearrange the function to make x the subject.	$y = x^2$	$x = \sqrt{y}$
2.	Convert x coordinates to y coordinates.	$y(1) = 1^2 = 1$ $y(3) = 3^2 = 9$	
3.	Integrate the function with respect to $\mathrm{d}y$.	$\pi\int_1^9 \sqrt{y}^2\,\mathrm{d}y = \pi\int_1^9 y\,\mathrm{d}y$ $= \pi\left[\frac{1}{2}y^2\right]\Big\|_1^9$ $= \pi\left(\frac{1}{2}\times 81 - \frac{1}{2}\times 1\right) = 40\pi$	

To find volume of revolution between two graphs, use the following formula (works the same way with $\mathrm{d}y$):

$$V = \int_a^b \pi\left[(\text{Outer radius})^2 - (\text{Inner radius})^2\right]\mathrm{d}x$$

6.3 Differential equations

An ordinary differential equation (ODE) is an equation that relates functions of an independent variable and its derivatives. Differential equations are really important in mathematics as they allow to solve a lot of real world and science problems. There are several methods for solving the differential equations, including analytical approaches and numerical methods.

6.3.1 Setting up differential equations

Questions may ask you to set up a differential equation. This is all about linking the rate of change of a variable, $\frac{\mathrm{d}y}{\mathrm{d}t}$, to the variable itself, y.

Example.

The growth of an algae, G, at time t, is proportional to $\sqrt{G}$. Find a differential equation to model this.

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sqrt{G}$$

The rate of change of salt concentration S in a tank at time t is proportional to $4St$. Find a differential equation to model this.

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 4St$$

6.3.2 Separation of variables

One of the common approaches to solve an ODE is to use method of separation of variables. It can be used when one is able to separate all y dependent terms to the left and all x dependent terms to the right. Then it is possible to solve each side in relation to their own variables.

If no initial conditions are given then an ODE has infinite solutions. A general solution can be found, with constants, to represent this.

Solving an ODE using separation of variables

Solve $\frac{dy}{dx} = 2xy$ which satisfies the initial condition $y(0) = 2$.

1.	Identify that the ODE can be solved using separation of variables and move all y and x dependent terms to their respective sides.	$\frac{dy}{dx} = 2xy$ $\frac{dy}{y} = 2x\,dx$
2.	Integrate each side separately. Remember to add a constant on the x side.	$\int \frac{dy}{y} = \int 2x\,dx$ $\ln(y) = x^2 + C$
3.	Solve for y.	$y = e^{x^2+C} = ke^{x^2}$ This is the general solution.
4.	Plug in initial conditions to find the value of the constant.	$y(0) = 2 = ke^0 = k$ $y(x) = 2e^{x^2}$

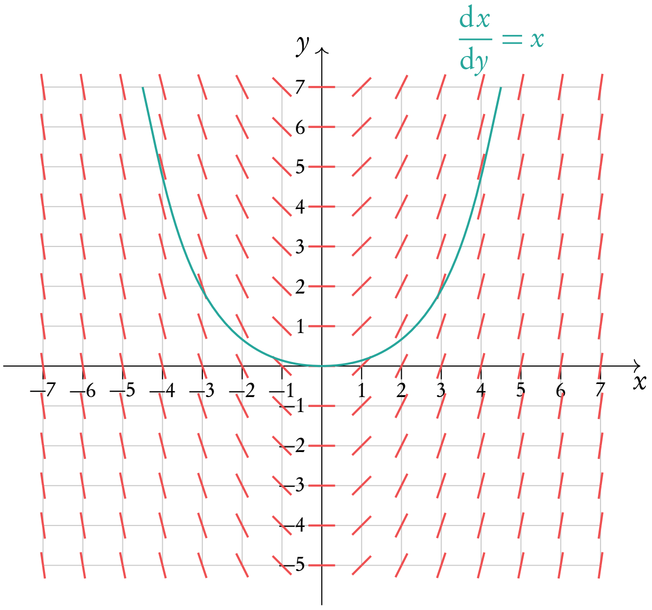
6.3.3 Slope fields

The many solutions to ODEs can be displayed on a set of axes. Small lines are drawn for many points on the axes to represent the gradient of the solution. Different solutions, which correspond to different initial conditions, can be drawn following the curve of the small lines.

At the point $(1,2)$ the gradient is $\frac{dy}{dx} = x = 1$. A short line of gradient 1 is drawn at that point.

At the point $(5,7)$ the gradient is $\frac{dy}{dx} = x = 5$. A short line of gradient 5 is drawn at that point.

A specific solution can be drawn onto the slope field. Pick $y(0) = 0$ as the initial condition and the solution can be drawn as on the slope field.



6.3.4 Euler's method

There are a lot of differential equations that are hard or impossible to solve analytically. Yet we still would like to know the value of a function given its initial conditions. Then we have to use numerical methods to solve the differential equation. Euler's method is a simple method that allows to find a value of a function given its first derivative and initial conditions. It is easy to derive the Euler's method from the approximation of a derivative at a point:

$$y'(x) = \frac{y(x+h) - y(x)}{h}$$

Solving for $y(x+h)$ gives us:

$$y(x+h) = y(x) + y'(x)h$$

Where we can rename for recursive purposes $y(x+h) = y_{n+1}$, $y(x) = y_n$ and $y'(x) = f(x_n, y_n)$, where $f(x_n, y_n)$ is our differential equation:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Now knowing initial conditions and using a small step h, it is possible to find an approximate value of y at required point x. The smaller step h, the more accurate is the solution. But it also takes longer to calculate it. It is recommended to create a table with all required variables to keep it clean and orderly.

Example.

Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem $\frac{dy}{dx} = \sin(x+y)$, $y(0) = 1$ at $y(0.5)$

n	x_n	y_n	$f(x_n, y_n)$
0	0.0	1	0.8415...
1	0.1	$1.0000 + 0.1 \times 0.8415 = 1.0842$	0.9262...
2	0.2	$1.0842 + 0.1 \times 0.9262 = 1.1768$	0.9812...
3	0.3	$1.1768 + 0.1 \times 0.9812 = 1.2749$	1.0000...
4	0.4	$1.2749 + 0.1 \times 1.0000 = 1.3749$	0.9792...
5	0.5	$1.3749 + 0.1 \times 0.9792 = 1.4728$	

Thus $y(0.5) \approx 1.47$. A computer gives numerical solution equal to $y(0.5) = 1.47825\ldots$ So we are about 0.01 off the actual answer. That is pretty good for such a simple numerical method!

Euler's method can also be used to find a numerical solution for coupled systems. In coupled systems, the rate of change of y and x both rely on functions of x, y, and time t.

$$\frac{dx}{dt} = f_1(x, y, t) \qquad \frac{dy}{dt} = f_2(x, y, t)$$

Euler's method for these equations is similar to the previous method.

$$\begin{aligned} x_{n+1} &= x_n + h \times f_1(x_n, y_n, t_n) \\ y_{n+1} &= y_n + h \times f_2(x_n, y_n, t_n) \\ t_{n+1} &= t_n + h \end{aligned}$$

Use Euler's method with step size $h = 1$ to approximate the solution to the coupled initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + y^3 + t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + 5y + t$, $y(2) = 0$, $x(2) = 0$, at $t = 4$.

A similar table can be set up. The variable t_n only depends on the step size, so this is put first. The first set of values is given by the initial condition. From them the values of $f_1 = x^2 + y^3 + t$ and $f_2 = 2x + 5y + t$ can be calculated. Then we move onto the second line, and use the equations in Euler's method to calculate t_n, x_n, and y_n. The process is repeated until we reach $t = 4$.

n	t_n	x_n	y_n	$f_1(x_n, y_n, t_n)$	$f_2(x_n, y_n, t_n)$
0	2	0	0	2	2
1	2.5	$0 + 0.5 \times 2 = 1$	$0 + 0.5 \times 2 = 1$	4.5	9.5
2	3	$1 + 0.5 \times 4.5 = 3.25$	$1 + 0.5 \times 9.5 = 5.75$	203.67	38.25
3	3.5	$3.25 + 0.5 \times 203.67 = 105.01$	$5.75 + 0.5 \times 38.25 = 110.00$	1342031	763.52
4	4	$105.01 + 0.5 \times 1342031 = 671121$	$110 + 0.5 \times 763.52 = 491.76$		

Thus we have $x(4) = 671121$ and $y(4) = 491.76$.

Sometimes coupled systems are not in the form as seen here. In order to perform Euler's method on them they must be first rearranged, often by introducing a new variable.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t)$$

In this case we set $\frac{\mathrm{d}x}{\mathrm{d}t} = y$, noticing that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}y}{\mathrm{d}t}$. We then have the following two coupled equations to use Euler's method on.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = f(x, y, t)$$

Example.

Finding two coupled equations from the following second order (containing a second derivative) differential equation by substituting.

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = x^2 + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5t^3$$

Let $\frac{\mathrm{d}x}{\mathrm{d}t} = y$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x^2 + 2y + 5t^3$$

6.3.5 Coupled differential equations

Coupled equations can also be solved by hand, instead of using Euler's method. Coupled differential equations can be represented using a system matrix.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To solve the coupled differential equations, the eigenvalues and eigenvectors of the system matrix are input into the following solution formula. The eigenvectors are represented by $\mathbf{p}_1$ and $\mathbf{p}_2$. A and B are constants to be determined by initial conditions.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$$

Solving coupled differential equations

Solve the following coupled differential equations with initial conditions $y(0) = 1$ and $x(0) = 2$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -6x + 3y \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x + 5y$$

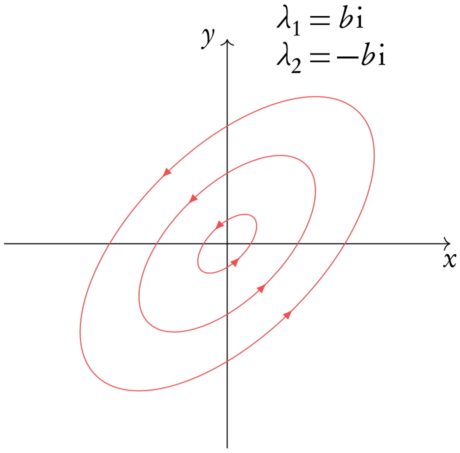
1.	Rewrite the equations in matrix format.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
2.	Find the eigenvectors and eigenvalues of the system matrix using your calculator.	$\lambda_1 = 6 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $\lambda_2 = -7 \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
3.	Fill in the eigenvectors and eigenvalues into the exact solution formula.	$\mathbf{x} = Ae^{6t}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + Be^{-7t}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$
4.	Use the initial conditions to find out the constants A and B.	$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = Ae^{6(0)}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + Be^{-7(0)}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} A \\ 4A \end{bmatrix} + \begin{bmatrix} -3B \\ B \end{bmatrix}$ $2 = A - 3B \qquad \& \qquad 1 = 4A + B$ $A = \frac{5}{13} \qquad \& \qquad B = -\frac{7}{13}$
5.	Fill in these values of A and B to give the final solution.	$\mathbf{x} = \frac{5}{13}e^{6t}\begin{bmatrix} 1 \\ 4 \end{bmatrix} - \frac{7}{13}e^{-7t}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

The system matrix can be used to draw a phase portrait, which shows the many solutions to the equations. Phase portraits are similar to slope fields, however they are constructed in a different way, using the two eigenvalues and eigenvectors of the system matrix.

Different eigenvalues correspond to the way in which the solutions move.

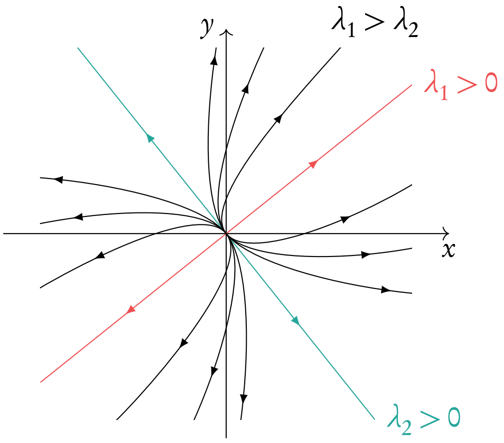
Eigenvalues: both imaginary

Solutions: form a circle or eclipse



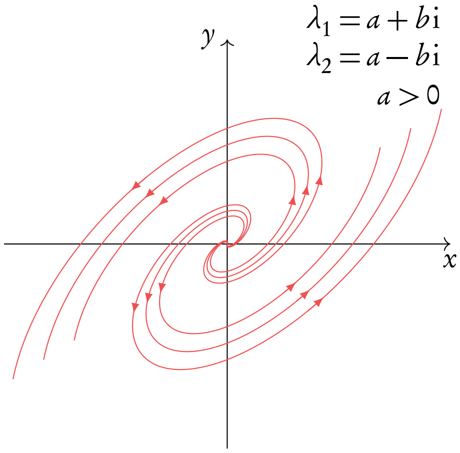
Eigenvalues: both real positive

Solutions: move away from the origin



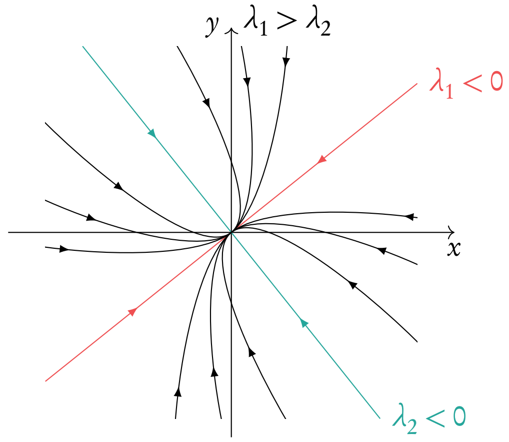
Eigenvalues: both complex with a positive real part

Solutions: move away from the origin in a spiral



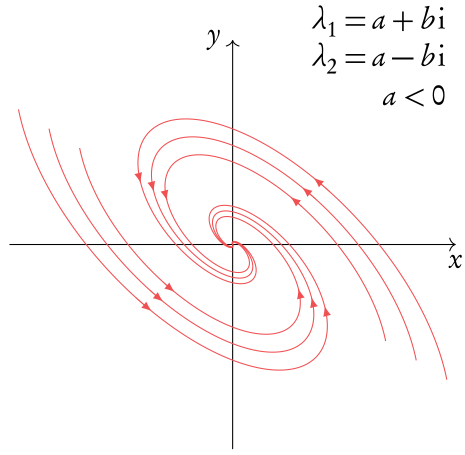
Eigenvalues: both real negative

Solutions: move towards the origin



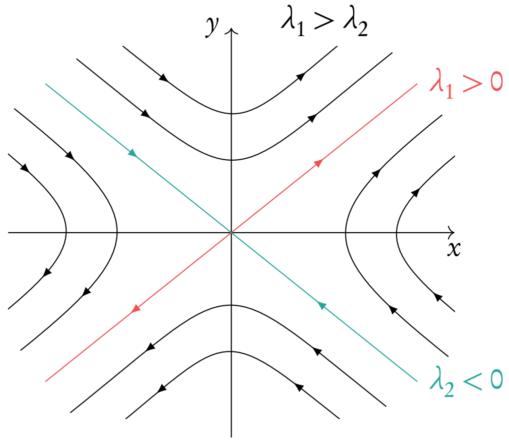
Eigenvalues: both complex with a negative real part

Solutions: move towards the origin in a spiral



Eigenvalues: one positive real and one negative real

Solutions: the origin is a saddle point



Drawing a phase portrait

Draw the phase portrait for the following coupled differential equations.

$$\frac{dx}{dt} = -6x + 3y \qquad \frac{dy}{dt} = 4x + 5y$$

1. Construct the system matrix and find the eigenvalues and eigenvectors.

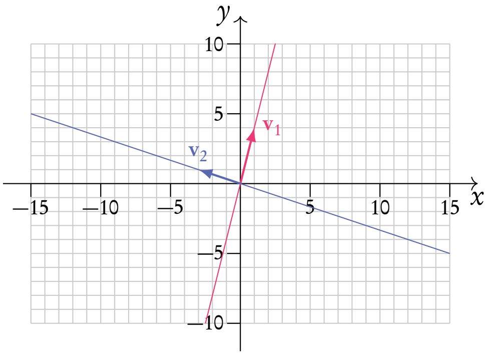
We found these in the previous example.

$$\lambda_1 = 6 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\lambda_2 = -7 \qquad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

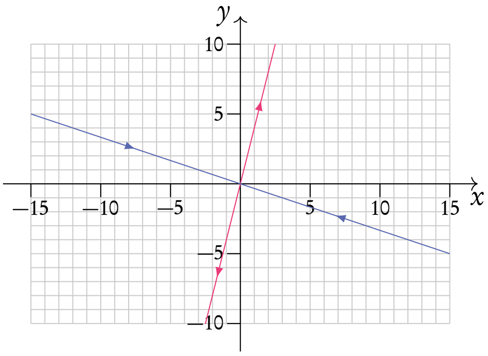
2. On a set of axes draw each eigenvector.

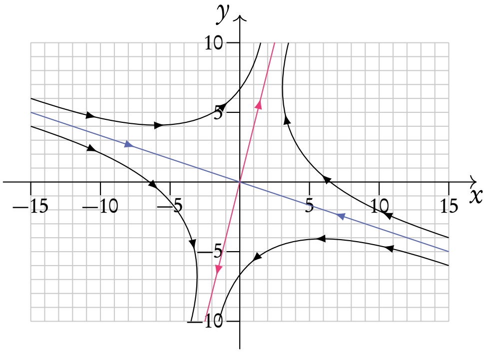
Extend lines past the ends of the vectors.



3. Determine which direction the extended lines point.

This depends on the eigenvalues associated with them. A negative eigenvalue indicates solutions moving towards the origin and a positive one indicates solutions moving away from the origin.



4.	Determine which eigenvalue/eigenvector will become more dominant as t approaches infinity.	This is always the larger eigenvalue. In this case $\lambda_1 > \lambda_2$.
5.	Draw on the solutions, beginning parallel to the eigenvector line with smallest eigenvalue and ending up parallel to the eigenvector line with largest eigenvalue.	Follow the directions of the arrows. 

PROBABILITY

7

Table of contents & cheatsheet

Definitions

Sample space the list of all possible outcomes.

Event the outcomes that meet the requirement.

Probability for event A, $P(A) = \dfrac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$.

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.

Conditional probability the probability of A, given that B has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

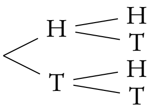
7.2. Multiple events 120

Probabilities for successive events can be expressed through tree diagrams or a table of outcomes.

Table of outcomes

	H	T
H	H,H	H,T
T	T,H	T,T

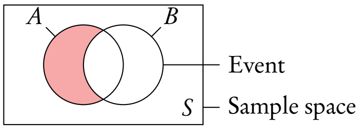
Tree diagram



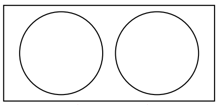
In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

7.1. Single events 118



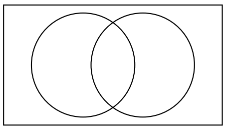
Mutually exclusive



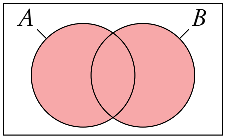
$P(A \cup B) = P(A) + P(B)$
$P(A \cap B) = 0$

Combined events

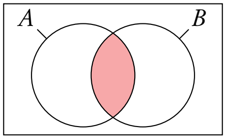
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$



$A \cup B$ (union)



$A \cap B$ (intersect)



If independent: $P(A \cap B) = P(A) \times P(B)$.

Compliment, A' where $P(A') = 1 - P(A)$

Exhaustive when everything in the sample space is contained in the events

7.3. Distributions 122

For a distribution by function the domain of X must be defined as $\sum P(X = x) = 1$.

Expected value $E(X) = \sum x P(X = x)$

Binomial distribution $X \sim B(n, p)$ used in situations with only 2 possible outcomes and lots of trials

On calculator:

- `Binompdf(n,p,r)` $P(X = r)$
- `Binomcdf(n,p,r)` $P(x \leq r)$

Mean $= np$ Variance $= npq$

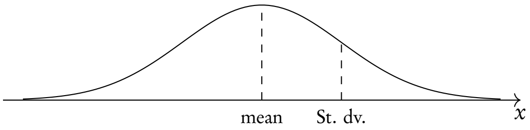
Poisson distribution $X \sim Po(m)$ used to show how many times an event is likely to occur in a time.

On calculator :

- `Poissonpdf(m,r)` $P(X = r)$
- `Poissoncdf(m,r)` $P(X \leq r)$

Mean $= m$ Variance $= m$

Normal distribution $X \sim N(\mu, \sigma^2)$



where $\mu =$ mean, $\sigma =$ standard deviation

On calculator:

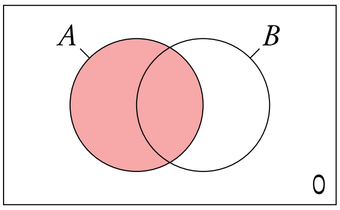
- `normcdf(lower bound, upper bound,` $= \mu$, σ`)`
- `invnorm(area,` $= \mu$, σ`)`

ACADEMY

7.1 Single events

7.1.1 Venn diagrams

Probability for single events can be expressed through Venn diagrams.



Sample space the list of all possible outcomes.

Event the outcomes that meet the requirement.

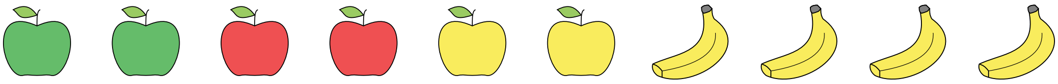
Probability for event A,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

Here the shaded circle.

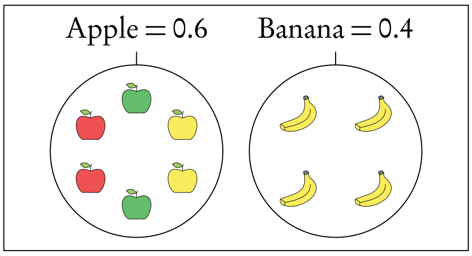
DB 4.5

Imagine I have a fruit bowl containing 6 apples and 4 bananas.



Example.

I pick a piece of fruit.
What is the probability of picking each fruit?

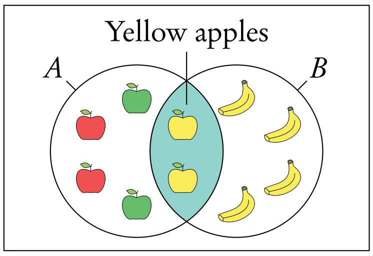


As apples cannot be bananas this is mutually exclusive, therefore $P(A \cup B) = P(A) + P(B)$ and $P(A \cap B) = 0$. It is also an exhaustive event as there is no other options apart from apples and bananas. If I bought some oranges the same diagram would then be not exhaustive (oranges will lie in the sample space).

DB 4.6

Example.

Of the apples 2 are red, 2 are green and 2 are yellow.
What is the probability of picking a yellow apple?



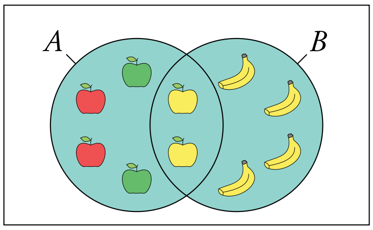
A: apples

B: yellow fruit

This is not mutually exclusive as both apples and bananas are yellow fruits. Here we are interested in the intersect $P(A \cap B)$ of apples and yellow fruit, as a yellow apple is in both sets $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

Example.

What is the probability of picking an apple or a yellow fruit?



A: apples

B: yellow fruit

This is a union of two sets: apple and yellow fruit.

The union of events A and B is:

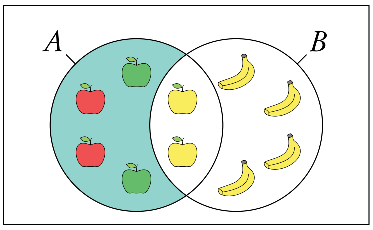
- when A happens;
- when B happens;
- when both A and B happen $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

When an event is exhaustive the probability of the union is 1.

DB 3.7

Example.

What is the probability of not picking a yellow fruit?



A: apples

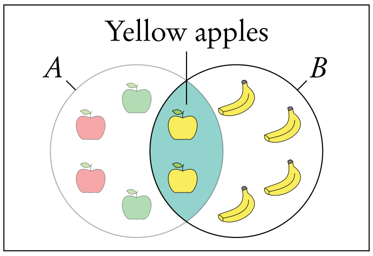
B: yellow fruit

This is known as the compliment of B or B'. $B' = 1 - B$.

Here we are interested in everything but the yellow fruit.

Example.

What is the probability of picking an apple given I pick a yellow fruit?

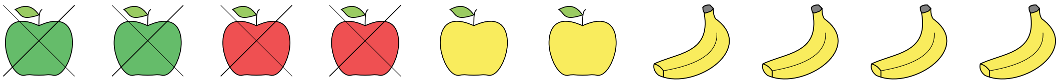


A: apples

B: yellow fruit

This is “conditional” probability in a single event. **Do not use the formula in the formula booklet.** Here we are effectively narrowing the sample space $= \dfrac{0.2}{(0.2+0.4)} = \dfrac{1}{3}$.

You can think of it like removing the non yellow apples from the fruit bowl before choosing.



Conditional probability $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

7.2 Multiple events

Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.
For independent events $P(A \cap B) = P(A) \times P(B)$

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

Conditional probability used for successive events that come one after another (as in tree diagrams). The probability of A, given that B has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Questions involving dependent events will often involve elements that are drawn “without replacement”. Remember that the probabilities will be changing with each new set of branches.

7.2.1 Tree diagrams

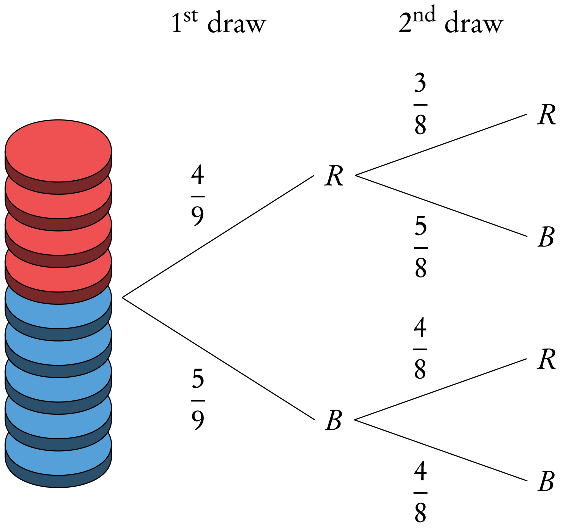
Probabilities for successive events can be expressed through tree diagrams. In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

Example.

Two disks are randomly drawn without replacement from a stack of 4 red and 5 blue disks. Draw a tree diagram for all outcomes.

The probability of drawing two red disks can be found by multiplying both probabilities of getting red $\left(\frac{4}{9} \times \frac{3}{8}\right)$.



R and B: $\frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$

R and B: $\frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

B and R: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

B and R: $\frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

The probabilities for each event should always add up to 1. The probabilities describing all the possible outcomes should also equal 1 (that is, the probabilities that we found by multiplying along the individual branches).

What is the probability to draw one red and one blue disk?

P(one red **and** one blue)

$(P(R) \text{ and } P(B))$ **or** $(P(B) \text{ and } P(R))$

$(P(R) \times P(B))$ $(P(B) \times P(R))$

$$\frac{20}{72} + \frac{20}{72} = \frac{40}{72} = \frac{5}{9}$$

It is common for conditional probability questions to relate to previous answers.

What is the probability to draw at least one red disk?

P(**at least** one red)

$$P(R \text{ and } R) + P(B \text{ and } R) + P(R \text{ and } B) = 1 - P(B \text{ and } B)$$

$$\frac{12}{72} + \frac{20}{72} + \frac{20}{72} = 1 - \frac{20}{72} = \frac{52}{72} = \frac{13}{18}$$

What is the probability of picking a blue disc given that at least one red disk is picked?

$$P(\text{blue disk} \mid \text{at least one red disk}) = \frac{P(\text{a blue disk})}{P(\text{at least one red disk})} = \frac{\frac{5}{9}}{\frac{13}{18}} = \frac{10}{13}$$

7.3 Probability distributions

7.3.1 Discrete random variables

Another way of representing multiple events is with sample space diagrams. These show all the possible combinations of events in a sample space. The probabilities of the possible outcomes are often summarised in a table.

Once tabulated we can use the probability distribution to find the expected value. It is best to think of this as the average value you would get if you repeated the action many times.

Probability distribution links each outcome of a statistical experiment with its probability of occurrence; usually presented as a table or function

Discrete random variable a variable measured from a random phenomenon that has a finite number of possible outcomes (e.g. the outcome of flipping a coin)

DB 4.7

Expected value $E(X)=\sum xP(X=x)$

Probability distributions

A fair coin is tossed twice, X is the number of heads obtained. Find the expected number of heads obtained on two throws of the coin.

1. Draw a sample space diagram.

	H	T
H	H, H	H, T
T	T, H	T, T

2. Tabulate the probability distribution.

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The sum of $P(X=x)$ always equals 1

3. Find the expected value of X: $E(X)$.

$$E(X)=\sum xP(X=x)$$
$$=0\cdot\frac{1}{4}+1\cdot\frac{1}{2}+2\cdot\frac{1}{4}=1$$

So if you toss a coin twice, you expect to get heads once.

7.3.2 Distribution by function

A probability distribution can also be given by a function.

The domain of X must be specified, as the sum of the probabilities must equal 1.

Probability distribution by function

$P(X=x)=k\left(\frac{1}{3}\right)^{x-1}$ for $x=1,2,3$. Find constant k.

1.	Use the fact that $\sum P(X=x)=1$ to set up an equation.	$k\left(\frac{1}{3}\right)^{1-1}+k\left(\frac{1}{3}\right)^{2-1}+k\left(\frac{1}{3}\right)^{3-1}=1$
2.	Simplify and solve for k.	$k+\frac{1}{3}k+\frac{1}{9}k=\frac{13}{9}k=1$. So, $k=\frac{9}{13}$.

7.3.3 Binomial distribution

Binomial distribution type of probability distribution used to calculate the probability of obtaining a certain number of successes in a given number of trials

Binomial distribution is used in situations with only 2 possible outcomes (e.g. success or failure) and lots of trials.

Using GDC

In your exam you will be expected to find probabilities from binomial distributions using your GDC. There are two different functions that you can use for this. For both you will need to know the number of trials (n), the probability of success (p) and the expected number of successes (r).

Use `Binompdf` (n, p, r) for questions asking for the probability of an exact outcome, $P(X=r)$.
Use `Binomcdf` (n, p, r) for questions asking for the probability of a range of consecutive values, $P(X\leq r)$.

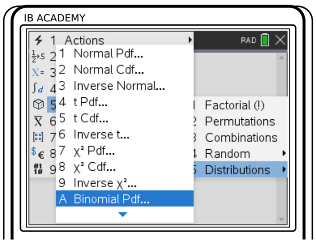
Note that by default `Binomcdf` only calculates $P(X\leq r)$ or in words "at most the value of r". Therefore you must remember to transform the function depending on the wording in the questions :

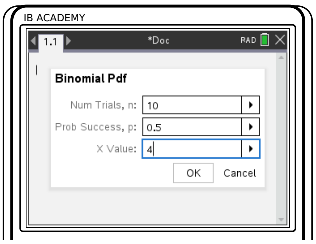
"Less than r" $P(X<r)=P(X\leq r-1)$
"More than r" $P(X>r)=1-P(X\leq r)$
"At least r" $P(X\geq r)=1-P(X\leq r-1)$

On some of the newer calculators you can specify what probability you are looking for, so this may not apply to you

Use binomial distribution to find a probability

A fair coin is flipped 10 times. Find the probability of flipping 4 heads.



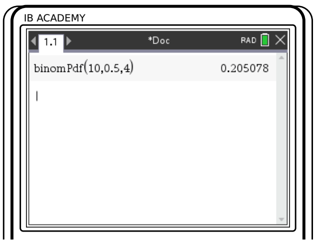


Binomial Pdf

Num Trials, n: 10

Prob Success, p: 0.5

X Value: 4



```
binomPdf(10,0.5,4)        0.205078
```

Press menu

`5: Probability`

`5: Distributions`

Select `Binompdf`

Enter values of

`n=10`

`p=0.5`

`r=4`

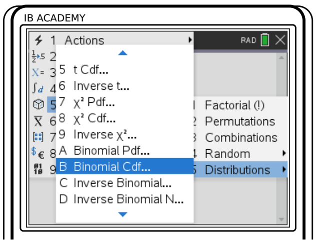
Press enter

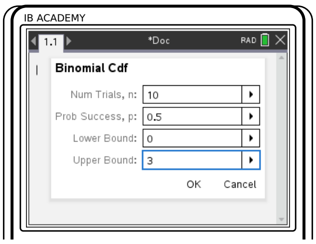
So the probability of getting exactly 4 heads is 0.205.

Use binomial distribution to find a probability

A fair coin is flipped 10 times. Find the probability of flipping less than 4 heads.

$P(X < 4) = P(X \leq 3)$





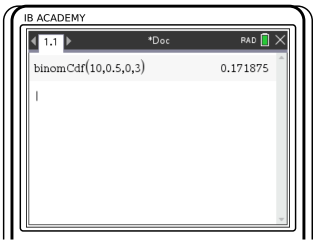
Binomial Cdf

Num Trials, n: 10

Prob Success, p: 0.5

Lower Bound: 0

Upper Bound: 3



```
binomCdf(10,0.5,0,3)        0.171875
```

Press menu

`5: Probability`

`5: Distributions`

Select `Binomcdf`

Enter values of

`n=10`

`p=0.5`

`lower bound=0`

`upper bound=3`

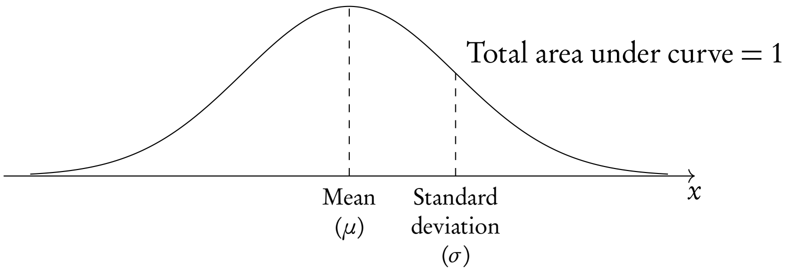
Press enter

So the probability of getting less than 4 heads is 0.172.

7.3.4 Normal distribution

A normal distribution is one type of probability distribution which gives a bell-shape curve if all the values and their corresponding probabilities are plotted. The bell shape is symmetrical around the mean, μ. The width of the bell shape is given by the standard deviation, σ.

The IB expects you to be aware that 68% of the data lies between μ and $\pm\sigma$ (the mean and 1 st.dev. either side of it, 95% lies between μ and $\pm 2\sigma$, and 99% lies between μ and $\pm 3\sigma$.



We can use normal distributions to find the probability of obtaining a certain value or a range of values. This can be found using the area under the curve; the area under the bell-curve between two x-values always corresponds to the probability for getting an x-value in this range. The total area under the normal distribution is always 1. This is because the total probability of getting any x-value adds up to 1 (or, in other words, you are 100% certain that your x-value will lie somewhere on the x-axis below the bell-curve).

Using GDC

Use your GDC to answer questions dealing with normal distributions. You will either need to find probabilities for given x-values or x-values for given probabilities. In both cases, you will need to know the mean (μ) and standard deviation (σ) for the given example. These will be given in the question.

Use `normalcdf` (lowerbound, upperbound, μ, σ) for the probability that x is between any 2 values.

> For lower bound $= -\infty$, use $-1E99$
>
> For upper bound $= \infty$, use 1E99

Even though you will be using your GDC, it's always useful to draw a quick sketch to indicate for yourself (and the examiner) what area or x-value you are looking for.

Use `invnorm` (ρ, μ, σ) to get an x-value for a given probability.

> Most calculators assume that ρ is to the *left* of x. When the area/probability you are given is to the right of x, subtract it from 1 to get the ρ to use in `invnorm`.

Finding a probability/ percentage/ the area under a normal distribution curve

The weights of pears are normally distributed with mean = 110 g and standard deviation = 8 g.
Find the percentage of pears that weigh between 100 g and 130 g

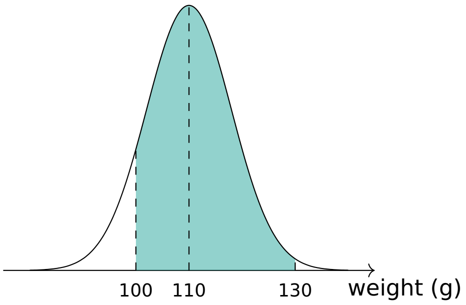
---

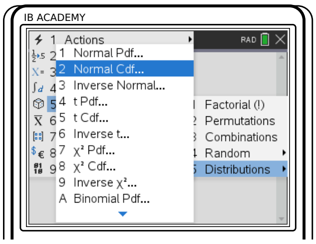
Sketch

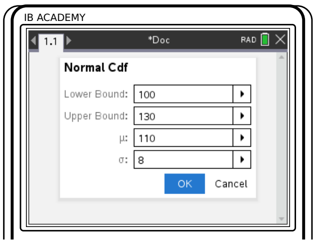
Indicate:

mean = 110 g
lower bound = 100 g
upper bound = 130 g

Shade area you are looking for

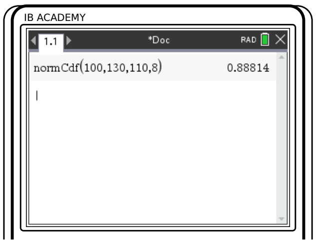






Normal Cdf

Lower Bound: 100
Upper Bound: 130
μ: 110
σ: 8



```
normCdf(100,130,110,8)          0.88814
```

Press menu, choose
`5: Probability`
`5: Distributions`
`2: Normal Cdf`

Enter lower and upper boundaries, mean (μ) and standard deviation (σ).
For lower bound = $-\infty$,
set `lower: -1E99`
For upper bound = ∞,
set `upper: 1E99`

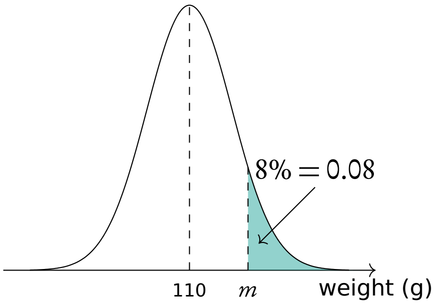
Press OK

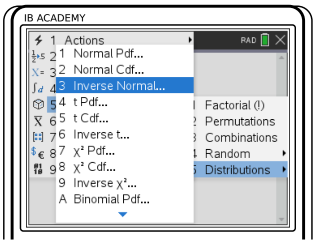
So 88.8% of the pears weigh between 100 g and 130 g.

Finding an x-value when the probability is given

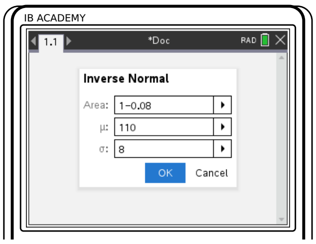
The weights of pears are normally distributed with mean $= 110$ g and standard deviation $= 8$ g. 8% of the pears weigh more than m grams. Find m.

Sketch





Press menu
`5: Probability`
`5: Distributions`
`3: Inverse Normal`



Inverse Normal

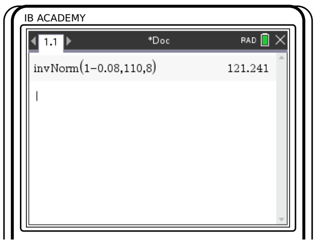
Area: 1−0.08

μ: 110

σ: 8

Enter probability (Area), mean (μ) and standard deviation (σ).
The calculator assumes the area is to the left of the x-value you are looking for.
So in this case:

$$\text{area} = 1 - 0.08 = 0.92$$



```
invNorm(1−0.08,110,8)        121.241
```

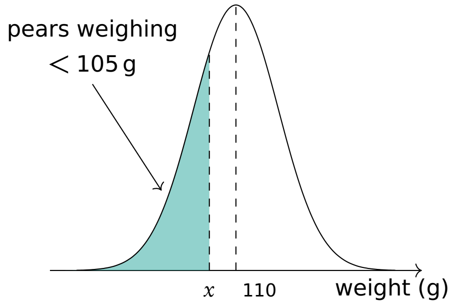
Press OK

So $m = 121$, which means that 8% of the pears weigh more than 121 g.

Finding an expected value

The weights of pears are normally distributed with $\mu = 110\,\text{g}$ and $\sigma = 8\,\text{g}$. 250 pears are weighed. Find the expected number of pears that weigh less than 105 g.

1. Sketch.



2. Use `normcdf` to find the relevant probability.

lower bound = -1E99
upper bound = 105
μ = 110
σ = 8 → ρ = 0.266

DB 4.8

3. Find expected value by multiplying the probability by the total number $E(x) = P(x) \times n$.

$$\text{expected number} = 0.266 \times 250$$
$$= 66.5$$

→ you would expect 67 pears to weigh less than 105 g

7.4 Poisson distribution

The Poisson distribution is used to calculate the probabilities of various numbers of “successes”. Each “success” must be independent. i.e. If mean number of calls to a fire station on weekday is 8. What is probability that on a given weekday there would be 11 calls?

Sometimes you are required to change the mean value, dependent on the problem. So if a mean number of “successes” is λ in period a min, then in period $c \times a$ min, there will be $c \times \lambda$ mean number of “successes”.

The sum of two independent Poisson distributions also follows a Poisson distribution.

Poisson distribution

The number of received calls by a hotel, can be modelled by Poisson distribution with a mean of 3.5 calls per minute.

a) Find probability that the hotel received at least 3 calls in each of the two consecutive minutes.

b) Find a probability that the hotel received exactly 15 calls in a random 5 minute interval.

1.	Determine expression for the required probability for a single occurrence.	$P($at least 3 calls in one minute$)$ $P(C>2)=1-P(C\leq 2)$
2.	Calculate probability using `poissoncdf` on GDC .	$1-\texttt{poissoncdf}(3.5,2)=0.67915$
3.	Raise the probability to the power of the number of occurrences.	Since we have two consecutive minutes, we square the probability: $0.67915^2=0.461$
4.	Adjust the mean to describe alternative parameters.	In part b) we are asked about a 5 minute interval: $3.5\times 5=17.5$
5.	Calculate using `poissonpdf` on GDC .	$\texttt{poissonpdf}(17.5,15)=0.0849$

STATISTICS

8

Table of contents & cheatsheet

Definitions

Population the entire group from which statistical data is drawn (and which the statistics obtained represent).

Sample the observations actually selected from the population for a statistical test.

Random Sample a sample that is selected from the population with no bias or criteria; the observations are made at random.

Discrete finite or countable number of possible values (e.g. money, number of people)

Continuous infinite amount of increments (e.g. time, weight)

Note: continuous data can be presented as discrete data, e.g. if you round time to the nearest minute or weight to the nearest kilogram.

8.2. Descriptive statistics 133

For 1 variable data with frequency use `1-Var Stats` on GDC.

Mean the average value

$$\bar{x} = \frac{\text{the sum of the data}}{\text{no. of data points}}$$

Mode the value that occurs most often

Median when the data set is ordered low to high and the number of data points is:

- odd, then the median is the middle value;
- even, then the median is the average of the two middle values.

Range largest x-value − smallest x-value

Variance σ^2 *calculator only*

Standard deviation $\sigma = \sqrt{\text{variance}}$ *calculator only*

Grouped data data presented as an interval

Use the midpoint as the x-value in all calculations.

Q_1 first quartile = 25th percentile

Q_2 median = 50th percentile

Q_3 third quartile = 75th percentile

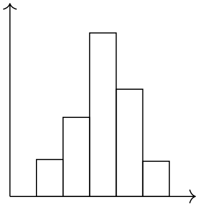
$Q_3 - Q_1$ interquartile range (IQR) = middle 50 percent

8.2.2. Statistical graphs 135

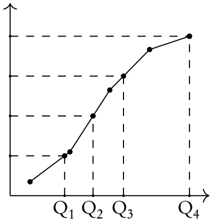
Frequency the number of times an event occurs in an experiment

Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

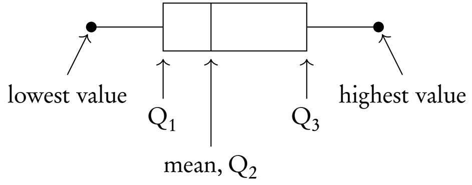
Histogram



Cumulative frequency



Box and whisker plot



8.3. Bivariate statistics 139

For analysis of data with two variables.
On GDC use `LinReg(ax+b)`.

Regression Line ($r = ax + b$)

Can be used to interpolate unknown data.

Interpretation of r-values

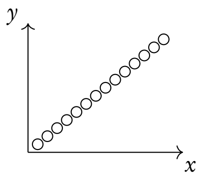
The correlation between the two sets of data. Can be positive or negative.

r-value	correlation
$0.00 \le \lvert r \rvert \le 0.25$	very weak
$0.25 \le \lvert r \rvert \le 0.50$	weak
$0.50 \le \lvert r \rvert \le 0.75$	moderate
$0.75 \le \lvert r \rvert \le 1.00$	strong

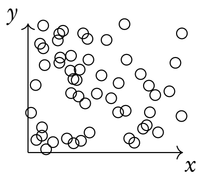
Correlation does not mean causation.

Scatter diagrams

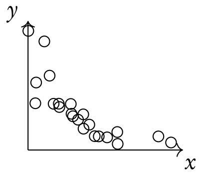
Perfect positive



No correlation



Weak negative



8.1 Basic statistical concepts

Population the entire group from which statistical data is drawn (and which the statistics obtained represent).

Sample the observations actually selected from the population for a statistical test.

Discrete finite or countable number of possible values. (e.g. money, number of people)

Continuous infinite amount of increments. (e.g. time, weight)

Note that continuous data can be presented as discrete data, e.g. if you round time to the nearest year or weight to the nearest kilogram.

8.1.1 Sampling techniques

In practice, samples can be obtained in different ways.

Simple random sample selected from the population with no bias or criteria; each member of the population has an equal chance of being picked

Convenience sample selected from a group that is easy to reach

Systematic sample selected at a fixed sampling interval; e.g. every 20th customer is asked to fill out a questionnaire

Stratified sample a population is first split into several mutually exclusive subgroups. A random sample is then drawn from each subgroup.

Quota sample like a stratified sample, except a non-random sample is drawn from each subgroup (e.g. a convenience sample)

Sampling bias when not all members of a population have an equal chance of being picked; can result in a sample not being representative of the population (e.g. convenience samples are likely to be biased)

8.2 Descriptive statistics

The mean, mode and median, are all ways of measuring "averages". Depending on the distribution of the data, the values for the mean, mode, median and range can differ slightly or a lot. They are all useful for understanding your data set.

Example data set: 6, 3, 6, 13, 7, 7 in a table:

x	3	6	7	13
frequency	1	2	2	1

DB 4.3

Mean the average value, $\bar{x} = \dfrac{\text{the sum of the data}}{\text{no. of data points}} = \dfrac{\sum x}{n} = \dfrac{\sum f x}{\sum f}$

Mode the value that occurs most often (highest frequency)

Median the middle value when the data set is ordered low to high. Even number of values: the median is the average of the two middle values. Find for larger values as $n + \frac{1}{2}$.

Range largest x-value − smallest x-value

Standard deviation indicates the spread of a data set, σ *calculator only*

Grouped data data presented as an interval; e.g. $10 \le x < 20\,\text{cm}$

- Use the midpoint as the x-value in all calculations. So for 10–20 cm use 15 cm.
- For 10–20 cm, 10 is the lower boundary, 20 is the upper boundary and the width is $20 - 10 = 10$.

Adding a constant to all the values in a data set or multiplying the entire data set by a constant influences the mean and standard deviation values in the following way:

	adding constant k	**multiplying by k**
mean	$\bar{x} + k$	$k \times \bar{x}$
standard deviation	σ	$k \times \sigma$

8.2.1 Quartiles

Q_1	first quartile The value of x so that 25% of all the data values are $\leq$ to it.	= 25th percentile.
Q_2	median	= 50th percentile
Q_3	third quartile	= 75th percentile
IQR	interquartile range $(Q_3 - Q_1)$	= middle 50 percent
Outlier	any point lower than $Q_1 - 1.5 \times \text{IQR}$ and larger than $Q_3 + 1.5 \times \text{IQR}$	

DB 4.2

Example.

Snow depth is measured in centimetres:
$30, 75, 125, 55, 60, 75, 65, 65, 45, 120, 70, 110.$
Find the range, the median, the lower quartile, the upper quartile and the interquartile range.

First always rearrange data into ascending order: $30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125$

1. The range:

$$125 - 30 = 95\,\text{cm}$$

2. The median: there are 12 values so the median is between the 6th and 7th value.

$$\frac{65+70}{2} = 67.5\,\text{cm}$$

3. The lower quartile: there are 12 values so the lower quartile is between the 3rd and 4th value.

$$\frac{55+60}{2} = 57.5\,\text{cm}$$

4. The upper quartile: there are 12 values so the lower quartile is between the 9th and 10th value.

$$\frac{75+110}{2} = 92.5\,\text{cm}$$

5. The IQR

$$92.5 - 57.5 = 35\,\text{cm}$$

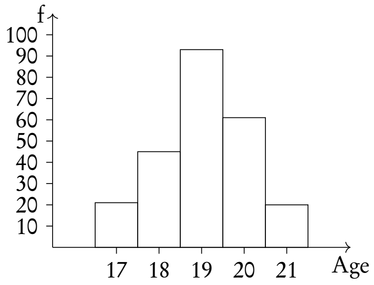
8.2.2 Presenting and interpreting data

Frequency the number of times an event occurs in an experiment

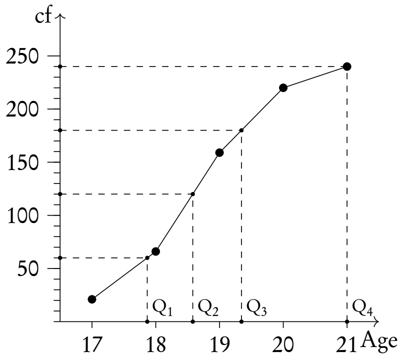
Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

Age	17	18	19	20	21
No. of students	21	45	93	61	20
Cumulative freq.	21	66	159	220	240

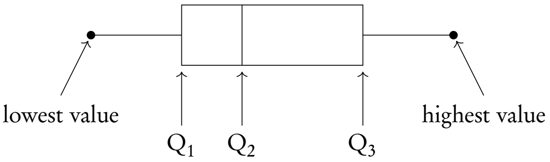
A cumulative frequency table summarises the cumulative frequencies for a data set.



A histogram is used to display and compare the frequencies for a specific condition. The frequencies (here: # of students) are displayed on the y-axis, and the different classes of the sample (here: age) are displayed on the x-axis. Neighbouring bars should be touching and their width should be drawn to scale (i.e. a wider class will be represented by a wider bar).



The cumulative frequency graph is used to display the development of the frequencies as the classes of the event increase. The graph is plotted by using the sum of all frequencies for a particular class, added to the frequencies for all the classes below it. The classes of the event (age) are displayed on the x-axis, and the frequency is displayed on the y-axis. The cumulative frequency graph always goes upwards, because the cumulative frequency increases as you include more classes.

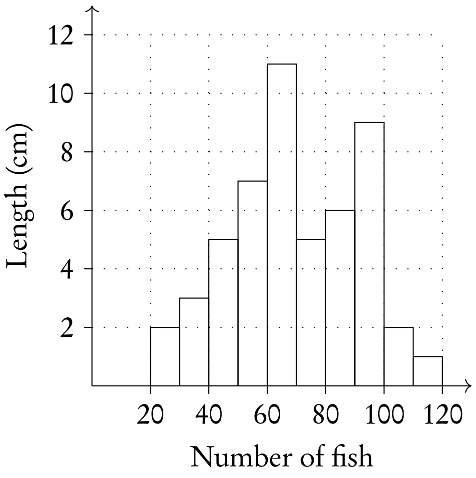


A box and whisker plot neatly summarises the distribution of a data set. It gives information about the range, the median and the quartiles. The first and third quartiles are at the ends of the box, the median is indicated with a vertical line inside the box, and the maximum and minimum points are at the ends of the whiskers. Box and whisker plots are drawn to scale in the x-direction

For your exam you will need to know how to find the value of Q_1, Q_2 and Q_3 using a cumulative frequency graph. First, determine the percentage of the quartile in question. Second, divide the total cumulative frequency of the graph (i.e. the total sample size) by 100 and multiply by the corresponding percentage. Then, you will have found the frequency (y-value) at which 25% for Q_1 / 50% for Q_2 / 75% for Q_3 of the sample is represented. To find the x-value, find the corresponding x-value for the previously identified y-value.

Example.

Using the histogram, create a cumulative frequency graph and use it to construct a box and whisker diagram.

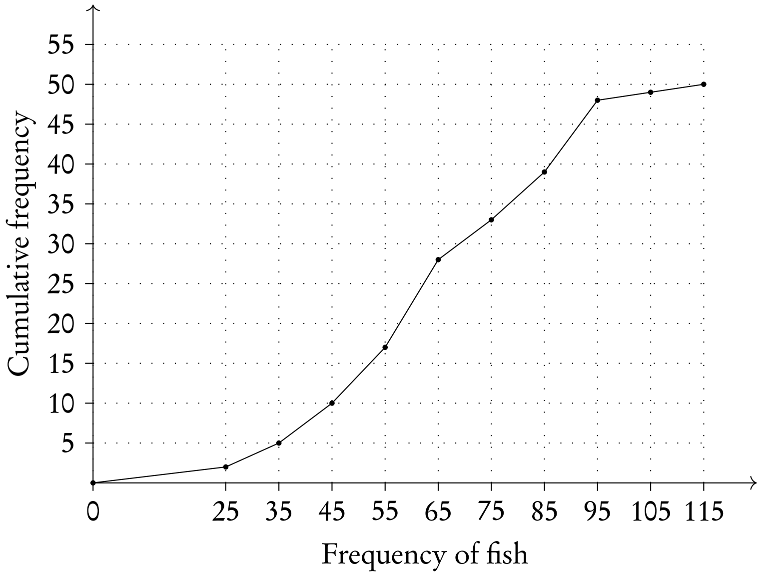


Write out the table for frequency and cumulative frequency.

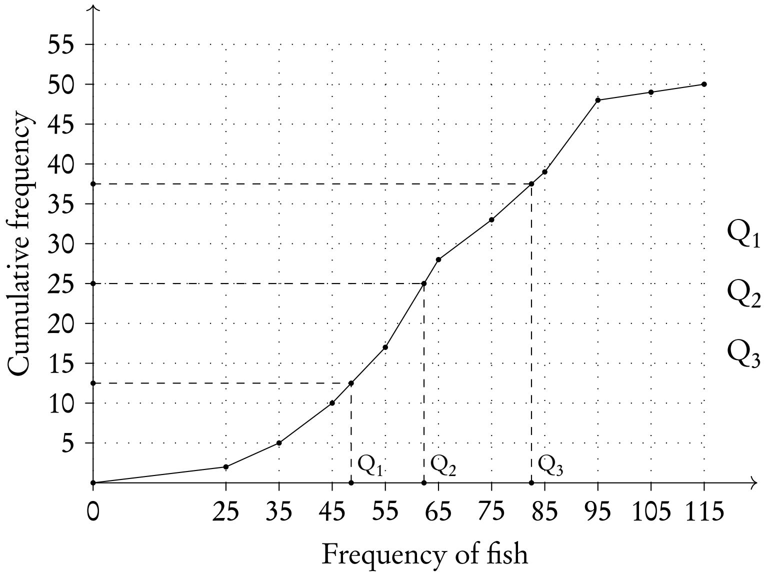
Frequency of fish	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Length of fish	2	3	5	7	11	5	6	9	1	1
Cumulative f.	2	5	10	17	28	33	39	48	49	50

Example.

Plot on cumulative frequency chart. Remember to use the midpoint of the date, e.g., 25 for 20–30.



Use graph to find Q_1, Q_2 and Q_3.

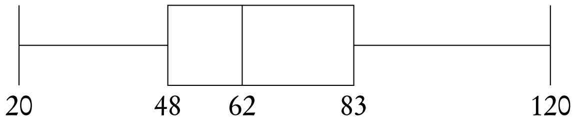


Q_1 25% of $50 = 12.5 \rightarrow 48$

Q_2 50% of $50 = 25 \rightarrow 62$

Q_3 75% of $50 = 37.5 \rightarrow 83$

Plot box and whiskers.

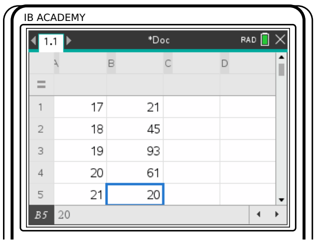


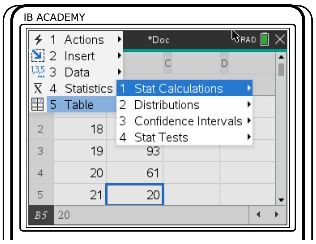
8.2.3 Using GDC

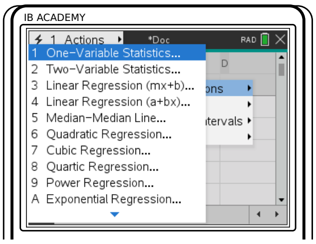
Finding the mean, standard deviation and quartiles etc.

Find the mean, standard deviation and Q1.

Age	17	18	19	20	21
No. of students	21	45	93	61	20



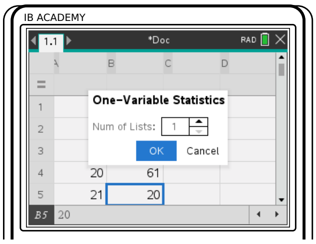


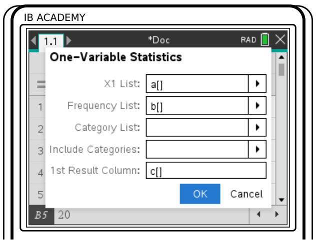


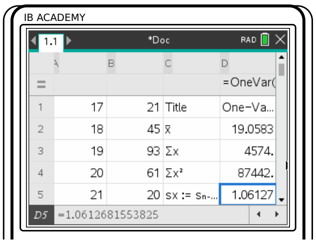
Press off on, go to `Lists and Spreadsheets.` Enter x-values in L1 and, if applicable, frequencies in L2

Press menu, choose `4: Statistics` `1: Stat Calculations`

`1: One-Variable Statistics`







Enter `Num of lists: 1.`

Press OK

Enter names of columns you used to enter your x-list and frequency list and column where you would like the solutions to appear: `a[]`, `b[]` and `c[]`.

Press OK

mean $= 19.1$

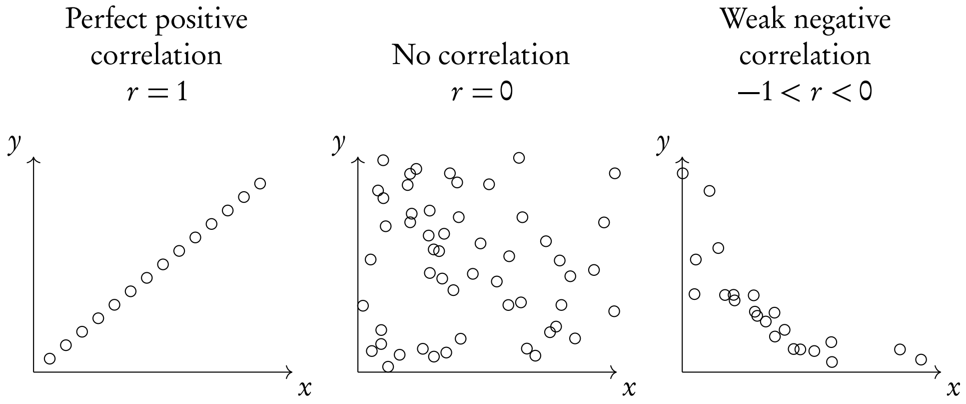
standard deviation $= 1.06$

Q1 $= 18.5$

8.3 Bivariate statistics

Bivariate statistics are about relationships between two different variables. You can plot your individual pairs of measurements as (x, y) coordinates on a scatter diagram. Analysing bivariate data allows you to assess the relationship between the two measured variables; we describe this relationship as correlation.

Scatter diagrams


Perfect positive correlation $r = 1$

No correlation $r = 0$

Weak negative correlation $-1 < r < 0$

Through statistical methods, we can predict a mathematical model that would best describe the relationship between the two measured variables; this is called regression. For your exam you will only have to focus on linear relationships, so only straight line graphs and equations. These so-called regression equations can be found using the GDC.

8.3.1 Pearson's correlation coefficient

Besides simply estimating the correlation between two variables from a scatter diagram, you can calculate values that will describe it in standardised ways. One of these is Pearson's correlation coefficient (r).

Pearson's correlation coefficient used to assess the strength of a linear relationship between two variables ($-1 \leq r \leq 1$)

$r = 0$ means no correlation.
$r = \pm 1$ means a perfect positive/negative correlation.

Interpretation of r-values:

r—value	$0 < \lvert r \rvert \leq 0.25$	$0.25 < \lvert r \rvert \leq 0.50$	$0.50 < \lvert r \rvert \leq 0.75$	$0.75 < \lvert r \rvert < 1$
correlation	very weak	weak	moderate	strong

Remember that correlation $\neq$ causation.

Calculate r while finding the regression equation on your GDC. Make sure that `STAT DIAGNOSTICS` is turned `ON` (can be found in the `MODE` settings), otherwise the r-value will not appear.

When asked to "comment on" an r-value make sure to include both, whether the correlation is:

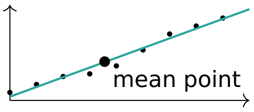
1. positive / negative
   and
2. strong / moderate / weak / very weak

Bivariate-statistics type questions

The height of a plant was measured the first 8 weeks

Week x	0	1	2	3	4	5	6	7	8
Height (cm) y	23.5	25	26.5	27	28.5	31.5	34.5	36	37.5

1. Plot a scatter diagram.



2. Use the mean point to draw a best fit line.

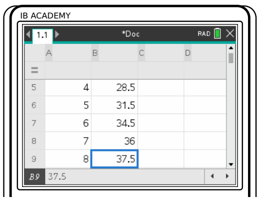
$$\bar{x} = \frac{0+1+2+\ldots+8}{9} = 3.56$$

$$\bar{y} = \frac{23.5+25+\ldots+37.5}{9} = 30$$

The line of best fit should pass through the mean point.

3. Find the equation of the regression line using GDC.

$y = 1.83x + 22.7$

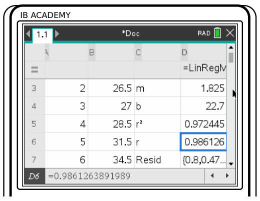


	A	B	C	D
=				
5	4	28.5		
6	5	31.5		
7	6	34.5		
8	7	36		
9	8	37.5		

B9 37.5

Press off on , got to “`Lists and Spreadsheets`”

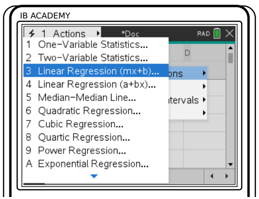
Enter x-values in one column (e.g A) and y-values in another column (e.g. B)



	A	B	C	D
=				=LinRegM
3	2	26.5	m	1.825
4	3	27	b	22.7
5	4	28.5	r²	0.972445
6	5	31.5	r	0.986126
7	6	34.5	Resid	{0.8,0.47...

D6 =0.9861263891989

So, equation of regression line is $y = 1.83x + 22.7$ and Pearson’s correlation

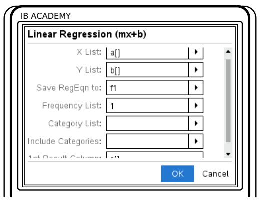


Press menu

`4: Statistics`

`1: Stat Calculations`

`3: Linear Regression (mx+b)`



Linear Regression (mx+b)

X List: a[]
Y List: b[]
Save RegEqn to: f1
Frequency List: 1
Category List:
Include Categories:

Enter

`X list: A [];`

`Y list: B[];`

`1st Result Column: C[]`

Press OK

4. Comment on the result.

Pearson’s correlation is $r = 0.986$, which is a strong positive correlation.

8.3.2 Spearman's rank correlation coefficient

Spearman's rank correlation coefficient used to assess the strength of a monotonic relationship between two variables ($-1 \leq r_s \leq 1$)

Monotonic relationship the values of two variables either both increase or both decrease, but not necessarily at a constant rate (i.e. can be linear or non-linear)

Spearman's rank correlation

Students' grades in Maths and Physics are recorded.

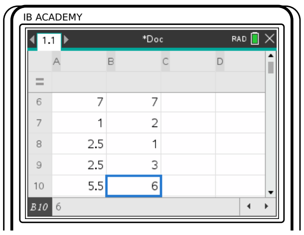
Maths	81	75	30	18	55	70	93	89	89	75
Physics	53	70	24	23	32	38	88	96	71	42

Find Spearman's Rank correlation coefficient and comment on it

1. Fill out a table of ranks for all the data. The ranks of tied values are the means of the ranks they would have had if they were different.

Maths	4	5.5	9	10	8	7	1	2.5	2.5	5.5
Physics	5	4	10	8	9	7	2	1	3	6

On TI-84, Spearman's Rank Coefficient can also be found using the RSX function



	A	B	C	D
6	7	7		
7	1	2		
8	2.5	1		
9	2.5	3		
10	5.5	6		

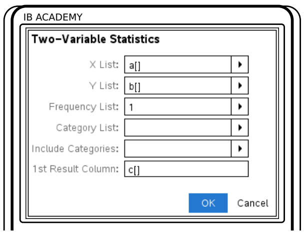
B10 6

Press +page doc

`4: Insert`

`6: Lists and Spreadsheets`

Enter table of ranks into two columns (e.g A and B)



Two-Variable Statistics

X List: a[]

Y List: b[]

Frequency List: 1

Category List:

Include Categories:

1st Result Column: c[]

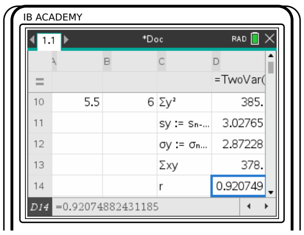
Press menu

`4: Stats`

`1: Stat Calculations`

`2: Two-variable Statistics`

Enter X and Y Lists



	A	B	C	D
=				=TwoVar(
10	5.5	6	Σy²	385.
11			sy := sn-...	3.02765
12			σy := σn...	2.87228
13			Σxy	378.
14			r	0.920749

D14 =0.92074882431185

Press OK

$r_s = 0.920749$

So there is a strong positive relationship between students' Maths and Physics grades.

8.4 Chi-squared test

8.4.1 Independence

Chi-square test of independence used to test independence of two variables

H_0 the variables are independent (null hypothesis)

H_1 the variables are **not** independent (alternative hypothesis)

Determine if the variables are independent by the χ^2 test

	Directors	Managers	Teachers	Totals
Male	26	148	448	622
Female	6	51	1051	1108
Totals	32	199	1499	1730

Perform a χ^2 test of independence at the 10% significance level to determine whether employment grade is independent of gender.

1. State the null and alternative hypotheses.

H_0: gender and employment grade are independent
H_1: gender and employment grade are not independent

2. Calculate the table of expected frequencies

Formula: $\frac{tf_1}{T} \cdot \frac{tf_2}{T} \cdot T$.

e.g. expected number of male directors:

$$\frac{622}{1730} \cdot \frac{32}{1730} \cdot 1730 = 11.5$$

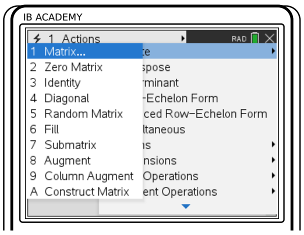
	Directors	Managers	Teachers
Male	11.5	71.5	539
Female	20.5	127.5	960

The expected frequency formula is not given in your data booklet, but you are expected to know it.

3. Write down the degrees of freedom df = (# rows - 1)(# columns - 1).

$\text{df} = (2-1)\cdot(3-1) = 2$

4. Write down the chi-square value using GDC..



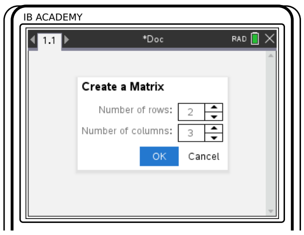
Enter data into GDC

Press menu

`7: Matrices & Vectors`

`1: Create`

`1: Matrix`

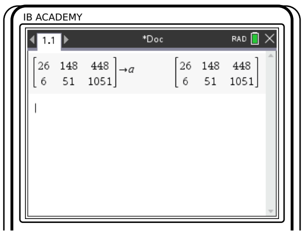


Number of rows: 2

Number of columns: 3

Enter dimensions of matrix to fit your data. Be sure you do *not* include the totals, so in this case you have a 2 × 3 matrix

Press OK



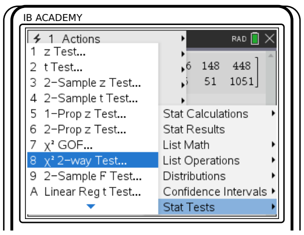
$$\begin{bmatrix} 26 & 148 & 448 \\ 6 & 51 & 1051 \end{bmatrix} \to a \qquad \begin{bmatrix} 26 & 148 & 448 \\ 6 & 51 & 1051 \end{bmatrix}$$

Enter the data as a matrix

press ctrl and sto→ var

Give matrix a name by typing it after the arrow (e.g. a)

press ≈ enter

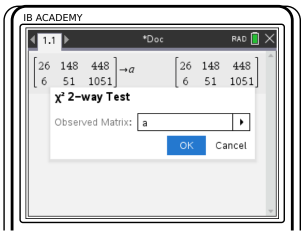


Press menu

`6: Statistics`

`7: Statistical Tests`

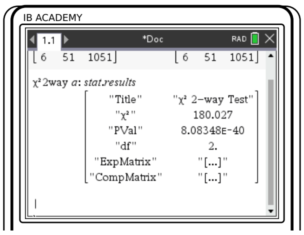
`8:` χ^2 `2-way Test`



Observed Matrix: a

Enter name of `Observed Matrix` (in this case a)

Press OK



```
χ² 2way a: stat.results
"Title"       "χ² 2-way Test"
"χ²"          180.027
"PVal"        8.08348E-40
"df"          2.
"ExpMatrix"   "[...]"
"CompMatrix"  "[...]"
```

So Chi-square value $\chi^2 = 180.03$, and p-value $= 8.08 \times 10^{-40}$

You will always be given the critical value in exam questions.

5. The critical value is 4.605. Use the critical value or the p-value to comment on your result.

Reject null hypothesis if:
χ^2 > critical value
or
p-value < significance level.

180.03 > 4.605
⟶ so we reject the null hypothesis; gender and employment grade are not independent.

8.4.2 Goodness of fit

Chi-square goodness of fit test used to determine whether categorical data fit a hypothesized distribution

H_0 the variables are consistent with the hypothesized distribution (null hypothesis)

H_1 at least one of the categories is **not** consistent with the hypothesized distribution (alternative hypothesis)

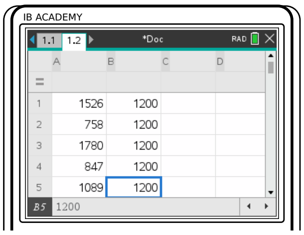
Determine goodness of fit of variables using the χ^2 test

Chocolate bar	Number sold
County	1526
Lars	758
Swix	1780
Silkyway	847
Trickers	1089
Total	6000

Perform a χ^2 test of goodness of fit at the 5% significance level to determine whether chocolate bar sales fit a uniform distribution.

1.	State the null and alternative hypotheses.	H_0: chocolate bar sales are uniformly distributed H_1: at least one of the chocolate bar sales does not fit a uniform distribution
2.	Find the expected frequencies.	Uniform distribution means that each category is equally likely. With 5 types of chocolate bar and 6000 sales recorded in total, the expected frequency for each category: $\frac{6000}{5} = 1200$
3.	Write down the degrees of freedom $\text{df} = (\#\text{ rows} - 1)$.	$\text{df} = (5 - 1) = 4$

4. Write down the chi-square value using GDC..



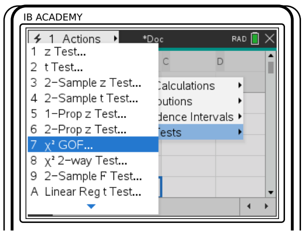
	A	B	C	D
=				
1	1526	1200		
2	758	1200		
3	1780	1200		
4	847	1200		
5	1089	1200		

B5 1200

Enter data into GDC

Press off on go to `Lists and Spreadsheets`

Enter observed and expected data in two columns (e.g. A and B)

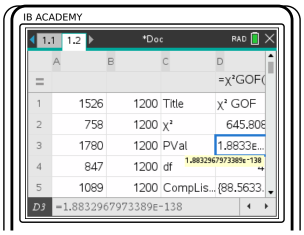


Press menu

`4: Statistics`

`4: Stat Tests`

`7:` χ^2 `GOF Test`



	A	B	C	D
=				=χ²GOF(
1	1526	1200	Title	χ² GOF
2	758	1200	χ²	645.808
3	1780	1200	PVal	1.8833E...
4	847	1200	df	4
5	1089	1200	CompLis...	{88.5633.

1.8832967973389E-138

D3 =1.8832967973389E-138

Enter name of `Observed` and `Expected` columns (in this case A and B)

Enter degrees of freedom

Press OK

So Chi-square value $\chi^2 = 645.808$, and p-value $=$ $4.440\,89 \times 10^{-16}$

5. Comment on your result.

$p < 0.05$

$\longrightarrow$ so we reject the null hypothesis; chocolate bar sales are not uniformly distributed.

8.5 T-test

The t-test is another type of statistical test that can be used to compare two groups. To apply a t-test, the variables you are testing should be normally distributed. Generally speaking, you can assume that this will be the case in exam questions that ask you to conduct a t-test.

Although there are several different t-tests (and different names for them!), we address only the type that you will need for your exam. This is also referred to as an Independent Samples t-test, Student t-test or unpaired t-test.

t-test tells you whether there is a significant difference between two groups by comparing their means

One-tailed test statistical significance is assessed only in one direction from a reference value (i.e. whether one mean is larger than the other *or* vice versa)

Two-tailed test statistical significance is assessed in both directions from a reference value (i.e. you test whether two groups are significantly different, but not in which direction)

Depending on what a question or task is asking of you, you will have to determine whether to use a one- or two-tailed test.

Conduct a two-sample t test

A popcorn producer wants to compare the effect of two different fertilisers on the number of usable corn kernels yielded. They measure the average number of usable kernels per corncob in a random sample taken from two batches, each treated with a different fertiliser.

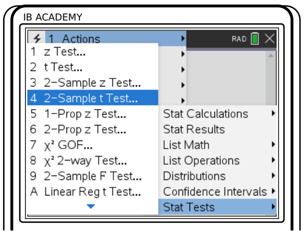
	Fertiliser A	Fertiliser B
Mean	768	803
Standard deviation	55	47
n	100	100

Use a t-test to determine whether the two fertilisers have a significantly different effect on corn kernel yield.

1. State the null and alternative hypotheses.

H_0: the means of kernels treated by fertiliser A and fertiliser B are equal ($\mu_1 = \mu_2$)
H_1: the means of kernels treated by fertiliser A and fertiliser B are not equal ($\mu_1 \neq \mu_2$)

2. Conduct t-test using GDC.

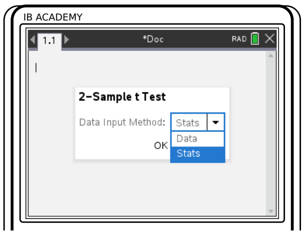


Press menu

`6:  Statistics`

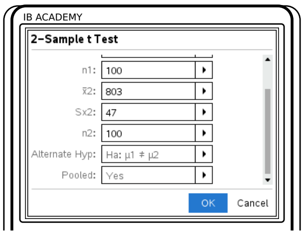
`7:  Stat Tests`

`4:  2-Sample t Test`



Select `Stats`.

OK



2−Sample t Test

n1: 100

x̄2: 803

Sx2: 47

n2: 100

Alternate Hyp: Ha: µ1 ≠ µ2

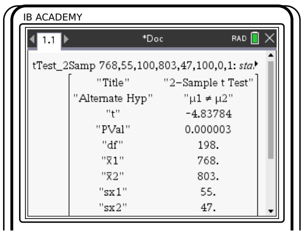
Pooled: Yes

Enter means ($\bar{x}$), standard deviations (Sx) and sample sizes (n)

Select appropriate H_1 (H_a)

Choose 'Yes' on Pooled

OK



```
tTest_2Samp 768,55,100,803,47,100,0,1: sta
           "Title"        "2−Sample t Test"
      "Alternate Hyp"        "µ1 ≠ µ2"
             "t"             -4.83784
           "PVal"            0.000003
            "df"              198.
            "x̄1"              768.
            "x̄2"              803.
           "sx1"               55.
           "sx2"               47.
```

So $t = 4.838$, and
p-value < 0.00001

As a general rule, you will always conduct a 'pooled' two-sample test. With a pooled test, we assume that the variance of the two samples is the same.

3. Comment on your result.

$p < 0.05$
→ so we reject the null hypothesis; the mean numbers of kernels yielded from batches treated by the fertilisers A and B are not equal.
Note that this two-tailed test does not tell us which fertiliser has a higher yield, just that there is a difference.

In your exam you will always be expected to use the p-value to compare the means of populations and draw conclusions from a t-test

8.6 Non-linear regression

Data which does not follow a linear relationship can be fitted with a non-linear regression curve. This is done using the method of least squares. The distance of each data point to a regression curve is analysed in order to find the best curve. Non-linear regression lines can be quadratic, cubic, exponential, power, or sine functions. Your calculator can do all of this for you.

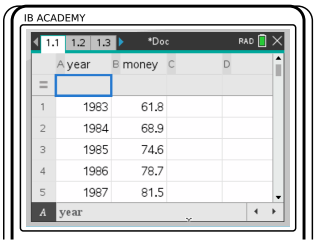
Once a regression has been calculated, the sum of square residuals (SS_{res}) can tell us how appropriate the curve is for our data. The larger the sum of square residuals, the less accurate the curve. This can also be done on your calculator.

Another indicator of how accurate the regression curve is, is the coefficient of determination (R^2 or r^2). This tells us how well the curve based on modelled data can replicate new observed data points. The higher the value of R^2, the better the curve is. For linear regression, r^2 is equal to the square of Pearson's correlation coefficient. Again, your calculator can find this.

Quadratic and cubic regression

Find a quadratic and a cubic regression curve for the data shown. Determine which is a better fit.

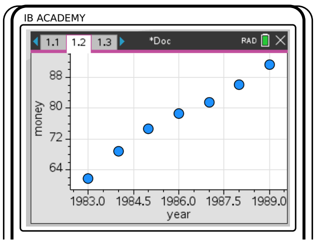
year	$ spent
1983	61.8
1984	68.9
1985	74.6
1986	78.7
1987	81.5
1988	86.1
1989	91.3



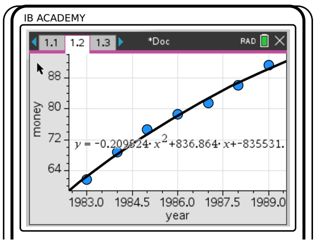
	A year	B money	C	D
=				
1	1983	61.8		
2	1984	68.9		
3	1985	74.6		
4	1986	78.7		
5	1987	81.5		

A year

Enter the data into your calculator in a table.



Make a scatter plot of it in a new window.



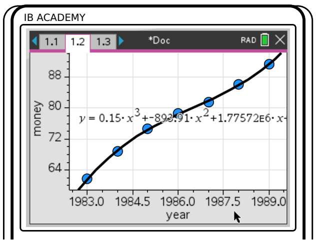
Plot the required regression curves.

Press menu

`4: Analyse`

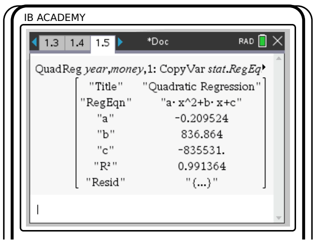
`6: Regression`

`4: Quadratic...`



...or

`5: Cubic`



```
QuadReg year,money,1: CopyVar stat.RegEq
  "Title"     "Quadratic Regression"
  "RegEqn"    "a·x^2+b·x+c"
  "a"         -0.209524
  "b"         836.864
  "c"         -835531.
  "R²"        0.991364
  "Resid"     "{...}"
```

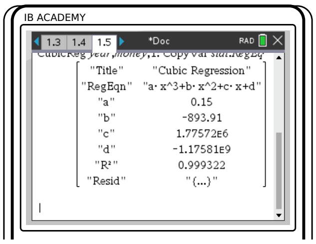
To find the R^2, open a new window and select calculator.

Press menu

`6: Statistic`

`1: Stat Calculations`

`6: Quadratic Regression...`



```
CubicReg year,money,1: CopyVar stat.RegEq
  "Title"     "Cubic Regression"
  "RegEqn"    "a·x^3+b·x^2+c·x+d"
  "a"         0.15
  "b"         -893.91
  "c"         1.77572E6
  "d"         -1.17581E9
  "R²"        0.999322
  "Resid"     "{...}"
```

...or

`7: Cubic Regression`

Then enter the data sets for the X and Y lists.

Look at the R^2 values you obtained. The curve with the higher R^2 value is the one you should pick. The cubic curve is a better regression curve for this data.

8.7 Linear transformations of random variables

Single random variables can be multiplied by a constant, or have a constant added to them to transform them. The following equations can be used to find the new expected value and variance of the transformed variable.

DB 4.14

$$\mathrm{E}(aX+b) = a\,\mathrm{E}(X)+b$$
$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

Independent random variables can also be added together to produce a new variable. The following equations are used to find the new expected value and variance.

DB 4.14

$$\mathrm{E}(a_1X_1 \pm a_2X_2 \pm \cdots \pm a_nX_n) = a_1\,\mathrm{E}(X_1) \pm a_2\,\mathrm{E}(X_2) \pm \cdots \pm a_n\,\mathrm{E}(X_n)$$
$$\mathrm{Var}(a_1X_1 \pm a_2X_2 \pm \cdots \pm a_nX_n) = a_1^2\,\mathrm{Var}(X_1) + a_2^2\,\mathrm{Var}(X_2) + \cdots + a_n^2\,\mathrm{Var}(X_n)$$

8.7.1 Unbiased estimators

When taking data often only a smaller sample of a larger population is examined. From this smaller sample the mean and variance of the entire population can be estimated. The unbiased estimate of the mean is referred to as $\overline{x}$ and the unbiased estimate of the variance is referred to as s_{n-1}^2. The actual mean of the entire population is μ and the actual variance of the entire population is σ^2.

The unbiased estimate of the mean is simply the mean of the sample.

$$\overline{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

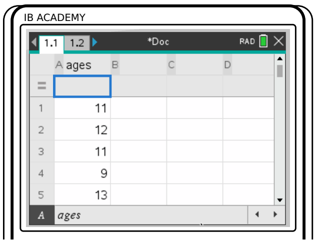
The unbiased estimate of the variance is given by the following equation.

$$s_{n-1}^2 = \frac{n}{n-1} s_n^2$$

Both unbiased estimators can be calculated using your calculator.

Unbiased estimators

A sample of 10 people is taken from the population of a school in Amsterdam, to measure their ages. The following data is obtained: 11, 12, 11, 9, 13, 10, 9, 13, 11, 9. Calculate unbiased estimates for the mean and standard deviation of the entire population.

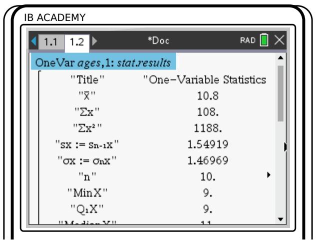


	A ages	B	C	D
=				
1	11			
2	12			
3	11			
4	9			
5	13			

Enter the data into a table on your calculator.

Open a new calculator page and go to one-variable statistics.

Press menu
`6: Statistics`
`1: Stat Calculations`
`1: One-Variable Statistics`
Select your data set and press OK



```
OneVar ages,1: stat.results
 "Title"        "One−Variable Statistics
 "x̄"            10.8
 "Σx"           108.
 "Σx²"          1188.
 "sx := sn-1x"  1.54919
 "σx := σnx"    1.46969
 "n"            10.
 "MinX"         9.
 "Q₁X"          9.
```

Read off the values from the table.

$\overline{x} = 10.8$ is both the sample mean and the unbiased estimate for the population mean. $sx = s_{n-1}x = 1.549$ is the unbiased estimate for the population standard deviation, and σx is the sample standard deviation.

8.8 Combining normal variables

Normal random variables can be added together to form a new variable. The new variable will also be normally distributed.

Suppose there is a large population which is normally distributed as follows.

$$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$$

A sample is taken from this population and the mean of it is calculated. This is repeated, such that the sample means form a distribution themselves, $\overline{X}$, as follows where n is the number of samples. The sample means form a normal distribution because the population follows a normal distribution.

$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

8.8.1 Central limit theorem

The sample means of a normal distribution will form a normal distribution, regardless of the number of samples.

A population is not normally distributed, but instead follows a different distribution. The mean of each sample taken from the population will begin to follow a normal distribution as the number of samples increases. This is what the central limit theorem states. In exams, if the number of samples is larger than 30, the sample mean can be considered normally distributed.

The central limit theorem means that the population distribution does not need to be known in order to perform tests on its mean – if enough samples are taken then we can use the normal distribution.

8.9 Confidence intervals of means

The mean of a population can be estimated using a sample, as seen in the unbiased estimators section. It is just the mean of the sample itself. However, this mean may not be entirely accurate and it can be useful to obtain a margin of error around it. Instead of claiming the mean to be 5, we can say the mean is 5 ± 0.2 with 95% confidence for example. This means that 95% of intervals made around sample means will include the population mean.

Confidence intervals can be calculated using two different distributions – the normal distribution and the t-distribution. The t-distribution is used when the population standard deviation, σ, is unknown – the sample standard deviation, s_n is used instead. The normal distribution (z-distribution) is used when the population standard deviation is known.

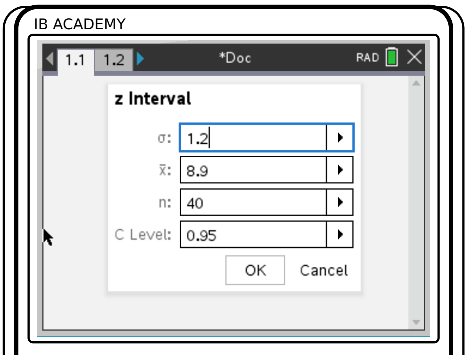
Calculating confidence intervals

A sample of 40 goldfish is taken from a population and measured, with mean $\overline{x} = 8.9\,\text{cm}$. The population standard deviation is $\sigma = 1.2\,\text{cm}$. Estimate, with 95% confidence, the mean length of the entire goldfish population.

1. Decide which distribution should be used.

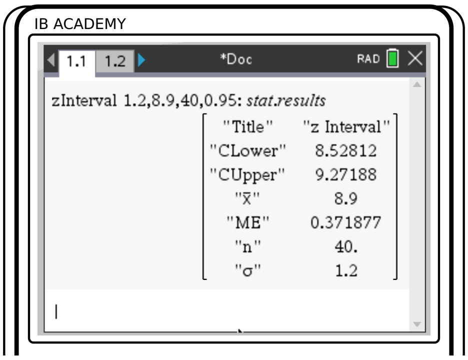
The population standard deviation is known, so we pick the normal distribution (z-distribution).

2. Open a calculator page on your calculator and select the confidence interval option.



z Interval

σ: 1.2
x̄: 8.9
n: 40
C Level: 0.95



```
zInterval 1.2,8.9,40,0.95: stat.results
 "Title"   "z Interval"
 "CLower"   8.52812
 "CUpper"   9.27188
 "x̄"        8.9
 "ME"       0.371877
 "n"        40.
 "σ"        1.2
```

Press menu
`6: Statistics`
`6: Confidence Intervals`
`1: z Interval`
Select `stats` as the data input method if you are given the mean and variance.
Select `data` as the data input method if you are given a set of numbers.

Press OK and read off your confidence interval bounds.
`CLower` is the lower bound and `CUpper` is the upper bound.
Our confidence interval is between 8.53 cm and 9.27 cm,
which means we have
$\mu = 8.9\,\text{cm} \pm 0.37\,\text{cm}$.

3. Write a sentence to explain the confidence interval.

With 95% confidence, the average length of goldfish in the goldfish population is $8.9\,\text{cm} \pm 0.37\,\text{cm}$ (based on the sample data).

8.10 Critical values and regions

The null hypothesis is rejected if the test statistic falls within the *critical region*. The size of the critical region is equal to the significance level. The values at the boundary of the critical region are known as the *critical values*.

8.11 More tests

8.11.1 z-tests and t-tests

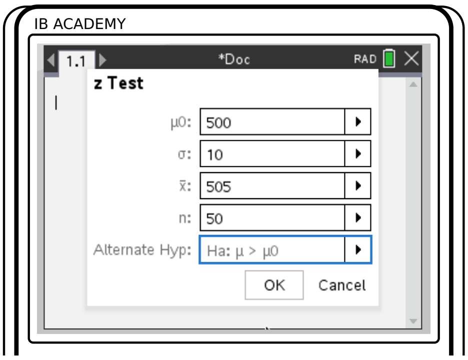
The normal test, or z-test, and t-test, are used to assess whether a sample mean is close enough to the supposed population mean. The z-test is used when the population standard deviation, σ, is known. The t-test is used when only the sample standard deviation, s_n, is known.

z-test and t-test

A machine fills bags of pasta with a labelled weight of 500 g. It is known the standard deviation of the bags filled by the machine is 10 g. To ensure the bags are filled to the correct weight, a sample of 50 bags is taken. The sample mean is found to be 505 g. Is the machine filling bags to the correct weight under a 5% significance level? Calculate the critical value and region for this test.

1.	Determine which test to use.	We know the standard deviation of the population so we pick the z-test.
2.	Write out the null and alternative hypotheses.	$H_0: \mu = 500$ $H_1: \mu > 500$

3. Use your calculator to find the p-value.



z Test

μ0: 500

σ: 10

x̄: 505

n: 50

Alternate Hyp: Ha: μ > μ0

Press menu

`6: Stats`

`7: Stat Tests`

`1: z Test`

$p\text{-value} = 0.000204$

4.	Accept the null hypothesis if the p-value is larger than the significance level.	$0.000204 < 0.05$ We reject the null hypothesis since our p-value is smaller than our significance level. The machine is not filling the bags correctly.
5.	Write down the distribution which the sample mean follows.	$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ $\overline{X} \sim \mathrm{N}\left(500, \frac{10^2}{50}\right)$
6.	Determine where the critical region is.	Let a be the critical value. The critical region is where the null hypothesis is rejected, so $P\left(\overline{X} > a\right) = 0.05$.
7.	Use the inverse normal distribution on your calculator to find the critical region and value.	We find the critical value is $a = \overline{x} = 503.29$, and the critical region is $\overline{x} > 503.29$.

8.11.2 Binomial test

The binomial test is used to assess whether observed test results differ from the expected results. For example, you expect 50% of people to like the colour blue and 10 people are asked if they like the colour blue. This is the observation, and only 2 out of 10 say that they like the colour blue. The binomial test assesses whether it is likely or not that 50% of the entire population like blue.

Binomial test

Is a normal die fair when 1 six is thrown in 30 throws? Test this at a 5% significance level. Rolling a six, X, follows a binomial distribution.

$$X \sim \text{Bi}(30, p)$$

1.	Set up the null and alternative hypotheses.	If the die is fair, then the probability of rolling a six is $\frac{1}{6}$. If the die is not fair then the probability is lower than $\frac{1}{6}$, since less sixes have been thrown than the expected number. $\text{H}_0\text{: } p = \frac{1}{6}$ $\quad$ $\text{H}_1\text{: } p < \frac{1}{6}$
2.	Calculate the probability of the scenario in the question occurring under the assumption that H_0 is true, using your calculator.	We want to find the probability of rolling 1 or less sixes. $X \sim \text{Bi}\left(30, \frac{1}{6}\right)$ $\quad$ $P(X \leq 1) = 0.029489$
3.	Accept the null hypothesis if this probability is larger than the significance level.	The probability of rolling 1 or less sixes is 2.95% so we reject the null hypothesis under the 5% significance level. The die is not fair.

8.11.3 Poisson test

A new model of machine in a factory is said to break down less than the previous model, which broke down 3 times per year on average. A sample of 4 new machines are found to break down in total 7 times across a year. The Poisson test assesses whether the new machines break down less often than the previous model.

Poisson test

An existing model of car is known to break down 4 times per year on average. A new model of car claims to break down less often.
20 randomly selected cars of the new model are found to break down a total of 56 times in a year. Test the claim at a 10% significance level.
The number of breakdowns per year for a car, X, follows a binomial distribution.

$$X \sim \mathrm{Po}(\lambda)$$

1.	Set up the null and alternative hypotheses.	H_0: $\lambda = 4$ H_1: $\lambda < 4$
2.	Find the expected mean for the sample size.	The sample is of 20 cars, so the expected number of breakdowns will be $4 \times 20 = 80$.
3.	Calculate the probability of the scenario in the question occuring, assuming H_0 to be true.	$X \sim \mathrm{Po}(80)$ $P(X \leq 56) = 0.002933$
4.	If this probability is smaller than the significance level then reject the null hypothesis.	The probability of having 56 or less breakdowns is 0.3% which is less than our significance level. The null hypothesis is rejected. The breakdown rate of cars has decreased.

8.11.4 Test for product moment correlation coefficient

The correlation coefficient, r, can be calculated for sets of bivariate data, which are samples from a larger population. The correlation coefficient for the entire population is written as ρ. The test for product moment correlation coefficients, or linear regression t test, tests the null hypothesis $\rho = 0$ against other alternative hypotheses $\rho > 0$, $\rho < 0$, or $\rho \neq 0$.

Test for product moment correlation coefficient

Employees at a workplace were randomly selected to provide the following data about the distance they live from their work and the number of times they arrived late in the past 4 weeks.

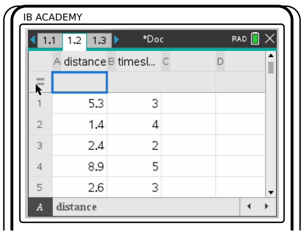
Distance (km)	Number of times late
5.3	3
1.4	4
2.4	2
8.9	5
2.6	3
1.3	1
0.7	3
5.8	6
8.7	8
10.2	6

Test at a 5% significance level whether there is a linear relationship between the two variables.

1. Write out the null and alternative hypotheses.

$H_0: \rho = 0$ $H_1: \rho \neq 0$

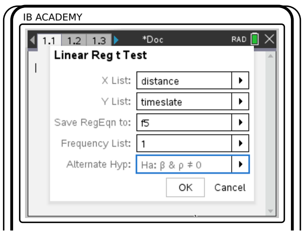
2. Use your calculator.



	A distance	B timesl...	C	D
1	5.3	3		
2	1.4	4		
3	2.4	2		
4	8.9	5		
5	2.6	3		

A distance

Enter the data into a table in your calculator.



Linear Reg t Test

X List: distance

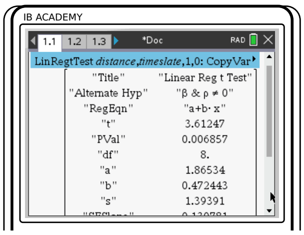
Y List: timeslate

Save RegEqn to: f5

Frequency List: 1

Alternate Hyp: Ha: β & ρ ≠ 0

Open a calculator page to perform the test.



```
LinRegtTest distance,timeslate,1,0: CopyVar
"Title"          "Linear Reg t Test"
"Alternate Hyp"  "β & ρ ≠ 0"
"RegEqn"         "a+b·x"
"t"              3.61247
"PVal"           0.006857
"df"             8.
"a"              1.86534
"b"              0.472443
"s"              1.39391
```

`6: Stats`

Press menu

`6: Statistics`

`7: Stat Tests`

`Linear Reg t Test (A)`

3. Reject the null hypothesis if the p-value is smaller than the significance level.

The p-value is 0.006857, which is smaller than the significance level, so we reject the null hypothesis. There is a linear relationship between the two variables.

8.12 Errors

When performing these statistical tests, errors are bound to happen. There are two types of errors.

Type I: H_0 is rejected when it is true
Type II: H_0 is accepted when it is false

The probability of making a type I error is equal to the significance level that the test is performed at.

Calculating the probability of making a type II error changes with different distributions.

In general the probability of making a type II error is

$$P\left(\text{accepting } \mathrm{H}_0 | \mathrm{H}_1 \text{ is true}\right)$$

which can also be written as

$$P\left(p\text{-value} > \text{significance level} | \mathrm{H}_1 \text{ is true}\right)$$

Example.

Poisson distribution type II errors

You are performing a Poisson test. Your null hypothesis is $\lambda = 5$, your alternative hypothesis is $\lambda < 5$.
If you observe a value of 6 or greater, you will reject your null hypothesis.
The actual λ is found to be equal to 4. What is the probability of making a type II error?

$$P\left(\text{accepting } \mathrm{H}_0 | \mathrm{H}_1 \text{ is true}\right)$$

We know $\lambda = 4$, and under this condition we need to see how likely it is that we accept the null hypothesis.
We accept the null hypothesis if we observe a value of 6 or greater.

$$P(X \geq 6 | \lambda = 4)$$
$$X \sim \mathrm{Po}(4)$$
$$P(X \geq 6) = 1 - P(X \leq 5) = 0.21487$$

The probability of a type II error is 21.5%.